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MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
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CHARLES GODFREY, M.V.O., M.A.

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CHARLES GODFREY, M.V.O., M.A.

CHARLES GODFREY was born in October, 1873. At an early age he went to King Edward's School, Birmingham, where he was head of the school, a position which depended entirely on a boy's ability in classics, and he spent at least five years in the top mathematical division under Rawdon Levett. The top mathematical division in those days consisted of a dozen boys, and generally contained three future major scholars of Trinity. Godfrey went up to Trinity as a major scholar in 1892, and was fourth wrangler in a brilliant year (1895), with Bromwich, Grace and E. T. Whittaker above him; in the following year he was placed in the first division of the first class of Part II. of the Mathematical Tripos; later on he won the Isaac Newton Studentship.

Between 1896 and 1899 he taught at Cardiff University, coached at Cambridge, and, for a year, helped Levett at Birmingham.

His real teaching career began when he went to Winchester in 1899. He soon became actively interested in the reform of mathematical teaching throughout the country. In 1901 he wrote a letter which was ultimately published as "The Letter from 23 Schoolmasters" (*Mathematical Gazette*, No. 31), which was the main cause of the appointment of the Mathematical Association Teaching Committee. From the first, though one of the three or four youngest members of the Committee, his influence was stronger than that of any other member; in some cases he was the power and stimulus behind the active work done by other members of the Committee, on other occasions he did the active work himself. A correspondent writes: "Those who served with him on the M.A. Committee will long remember at least two characteristics—his swift detection of a fallacy, and the perfect loyalty with which he accepted a decision on the few occasions on which he found himself in a minority."

I presume that it is as a text-book writer that his name is most widely known; but the writing of books was only a minor part of the work he did—far more important was the spade work he did to get reforms considered and to make them possible: he was driven almost reluctantly into writing books by the urgent need for them if reforms were to be carried through. He was a good mathematician, and was *persona grata* with the best mathematicians

at Cambridge and elsewhere ; added to this, he had enough experience in the actual teaching of schoolboys to grasp their difficulties and to appreciate the wrong-headedness of much of the mathematical teaching of a quarter of a century ago, and the difficulties under which an enlightened teacher had often to work.

The amount of work he did for the cause of mathematical teaching is really surprising. He was one of the most regular attendants at meetings of the M.A. Committee, and did much work between meetings on most of the reports of that Committee. He frequently read papers and spoke at annual meetings of the Mathematical Association, he addressed meetings of Preparatory School Headmasters, served on committees of the Headmasters' Conference, lectured in London and Oxford, inspected schools, examined for the certificate examinations of the Oxford and Cambridge Joint Board, London, Durham and Bristol Universities. He was one of Great Britain's representatives on the International Commission of Mathematicians. He was frequently called into consultation by examining bodies when mathematical schedules were under discussion—in such cases he carried great weight because of his wide experience, and because he was in touch with work much beyond school standards.

He was Senior Mathematical Master at Winchester from September, 1899, to April, 1905. In that time he completely revolutionised the old system ; the teaching was modernised and reorganised, a laboratory for practical mathematics was designed and started by him, and mathematics took a much better place than ever it had before in the general work of the school—all this was accomplished with no loss of scholarship, in fact the reverse was the case. His system still survives. An Old Wykehamist writes : " When he came to Winchester, he was, I think, regarded by the masses as one of prodigious and mysterious brains devoted to a subject very few knew anything about, but we who had anything to do with him respected him and liked him very much."

In 1905 he succeeded C. E. Ashford, another old pupil of Levett's, as Headmaster of the Royal Naval College at Osborne. There mathematics was necessarily of a much narrower range than at Winchester ; on the other hand, the subject had to be treated in a modern fashion and was free from the restraint of outside examinations. As headmaster he could use his very wide knowledge of other subjects—he had read very widely in literature, history and geography, he had a good knowledge of French and German, and he was much interested in music, and a lover of his garden and wild flowers.

In 1920 he became Professor of Mathematics at Greenwich, and at once threw himself into the teaching of advanced work there—it was surprising that all this work was just as fresh to him as it was when he left Cambridge twenty odd years before. He got into touch with the practical problems that confronted the naval officer, and brought his wide knowledge to bear in linking up the methods of the practical man with the work of the mathematician ; in that he started a great work which, it is to be hoped, will be continued.

He was not a man who pushed himself forward, and it is only those who knew him intimately who can appreciate how much of the reform of mathematical teaching in the last twenty years is really due to his influence. In private life he could talk with intimate knowledge and a dry humour on so many subjects that it was always a pleasure to be with him.

Right up to the time of his sudden illness he was so full of health and mental vigour and still had such a youthful freshness of appearance that he seemed likely to be a dominant influence on the mathematical teaching of the country for many years to come.

He died, from bronchial pneumonia after a short illness, on 4th April, 1924.

A. W. Siddons.

A PLEA FOR TEACHING PROBABILITY IN SCHOOLS.*

By W. Hope-Jones, B.A.

In your *Mathematical Gazette* for last month you read that "the mathematical theory of Probability is practically dead"—and that from a real authority on the subject whose name is known and honoured wherever statistics are analysed and laws deduced from them.

This is a cheering thought for a poor visionary whose ambition is to extend and popularise the study of Probability. But you will have noticed by now that the policies of all our three great political parties are dead and buried, according to the other two—and yet they carry on "business as usual"; and their courageous habit of ignoring trifles like death and burial has inspired my poor ghost to carry on as best it can with this phantom paper.

The relations of Probability, Statistical Inference, Psychology, and Metaphysics are at present in the melting-pot, and are the subject of much disagreement between those who understand them better than I do. I have nothing whatever to contribute to this discussion; in fact I intend to ignore it completely, except for expressing an earnest hope that it will not be allowed to come between the simple and obvious aspects of Probability and the simple and obvious type of student who is capable of studying them with profit and enjoyment. Relativity is all very well in its place: it begins to be dangerous if it cuts off the ordinary engineer from learning his ordinary Newtonian Dynamics; and I suggest that the higher controversial Probability stands in the same relation to the elementary Probability which you live by.

I can't tell you whether Probability exists or not. If it comes to that, I can't tell you whether you exist or not; but I am acting on the assumption that you do. And I believe that whether Probability exists or not, all of you order your lives very largely by it, more perhaps than you realise.

I have chosen the text "Why do you carry an umbrella?" to hang this sermon on, not because of the intrinsic importance of this particular problem, but because it is typical of many of life's greater problems. "Why do you insure against burglary or death?" "Why do you educate your children?" "Why do you discard one method of explaining Logarithms in favour of another?" "Why do you run for an unseen train?" "Why do you hurry through the Menin Gate?" and "What did you come to this meeting for?" Not one of these questions can be answered without some reference to Probability.

Now what about this umbrella? Before we start on it, let me remind you here of the idea called "Expectation." None of you knows what day of the week I was born on. Suppose I offer you a guinea if you guess it right, and you are to pay me half a guinea if you guess wrong. Then you have a chance of 1 in 7 of winning a guinea. Its value to you is $\frac{1}{7}$ of a guinea or 3 shillings. This is called your "Expectation of Gain." Your expectation of loss is $\frac{6}{7}$ of 10/6, because you have 6 chances out of 7 of losing 10/6. $\frac{6}{7}$ of 10/6 comes to 9 shillings. So your expectation of loss exceeds your expectation of gain by 6 shillings; and your resultant or total expectation of gain can be called *minus* 6 shillings.

Now come to the umbrella problem. Think of all the things that may happen to you in the day. You may get a little damp; you may get soaked; you may meet a distressed kitten in a pond, too far for your arm to reach, but accessible to the umbrella; you may repel with it the attack of a mad dog; you may get your umbrella stolen or punctured; or you may get put in gaol for poking out the bus-conductor's eye with it.

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* A paper read at the Annual Meeting of the Mathematical Association, January 8, 1924.

You estimate the value of all these advantages, taking in each case the difference between the advantages resulting from having your umbrella with you and from not having it, and counting the advantage as negative wherever you do better without it.

But when you have done that, it is useless to add up the whole lot and expect the resulting function of your circumstances to be of value in deciding for you the great question "Shall I carry my umbrella to-day?" It is useless because it gives equal weight to likely and unlikely events. Before you can solve the umbrella problem, you must multiply every advantage by its own probability, in order to get your expectation of that advantage, and then add up or integrate all your expectations. Integration is a better name for it than addition, because every intermediate degree of damage from rain is possible, from a limp collar to a death from pneumonia. And when you have estimated your integral, if its value is positive, then you decide to take your umbrella with you—but you may put it down to brush your coat and then forget it and go off without it.

Now all this sounds to you a gross exaggeration of your mental processes. You do not formulate your work like that. You have no unit in which you estimate advantages or expectations of advantages. But you do habitually compare lengths without troubling to express them in inches, and advantages and enjoyments without expressing them in any defined unit. And I suggest to you that this weighing of advantages against each other, multiplied by their respective probabilities so as to make them into expectations of advantage, is done instinctively by many people who are not familiar with the technical terms and professional notation. I want to commend to your attention the boy we read about two years ago in *Punch*, who, when in doubt, always used Subjunctives in his Latin Prose in preference to Indicatives, on the ground that "the old man always gets much wilder if you put Indicative where it ought to be Subjunctive than if you put Subjunctive where it ought to be Indicative." I don't suppose that boy had ever heard of "expectation of advantage," but obviously he understood the principles of it.

I take it that all of us consciously or unconsciously are engaged in trying to raise to a maximum the value of this function:

$$\iiiint (\text{Good})\, dx\, dy\, dz\, dt,$$

with no extra charge for more $\int$ signs if you believe in more dimensions—a function of the efforts of all of us, which can be differentiated partially with respect to any individual's actions.

(And here let me meet a possible objection. We all know that "four-and-twenty" may be the right name for the number of blackbirds baked in a pie, but $\underline{4|}$ a better name for the same number when it turns up as a denominator in a Binomial Coefficient.

They mean the same number; but each name is appropriate to the connection in which that number occurs.

So I don't want to incur the hostility of those who prefer to call this Integral by some other name, such as, for instance, "The Kingdom of Heaven." I am not putting up my Integral in opposition or rivalry to that conception, but only as the name for it which is most appropriate for the work I want to do with it now.)

"Good" probably means very much the same for all of us here; if not, this is not the right time to discuss what it means for one and what for another; but where the difference between one character and another comes in is in the choice of methods for increasing this function, and still more in the limits assigned to the integrals. For the most selfish, the limits are from I to me. To the boy who used Subjunctives when in doubt, the limits were from a whopping to a Latin Prose Prize. To the improvident, the time-integral

extends from now to about tea-time. When an occasional hero makes the range of all his Integrals from $-\infty$ to $+\infty$, an indignant world executes him or shuts him up as a criminal lunatic. But the point that concerns us most now is that in the very uncertain world we live in, this is an exceedingly discontinuous and badly-behaved function of our conduct; its partial differential coefficient is often incalculable, and we find ourselves in practice aiming at the maximum value of this function instead:

$$\iiiint \text{(Expectation of good)}\, dx\, dy\, dz\, dt,$$

every possible good being multiplied by its own probability of occurrence in order to obtain its expectation. And if I am right in believing that all the thoughtful people in the world are aiming at a maximum value of this Integral, I think you will agree with me that they would often do it the better for having a clearer conception of the end to which they have unconsciously been striving.

And that is why I am anxious that this bee of Probability which buzzes in my bonnet should buzz in yours as well. Not only this main problem of human life, but every little subdivision of it, "Shall I take my umbrella?" "Shall I try to join in this hymn that I don't know very well?" "Shall I put an Indicative or a Subjunctive?" "Shall I buy German toys?" involves the idea of Probability used consciously or unconsciously. My idea is that all these sub-problems would be better solved if the acquaintance with that idea were more direct and above-board. Even the hero of the Latin Prose story would have got more marks with numerical data to work on.

Now I believe that Probability has been relegated in the School curriculum to a dark and rather ridiculous corner, mainly because it has stopped short just where its chief interest begins. It has offered such answers as it can to "Yes and No" questions, "Will this happen or not?" "Is there an ace in his hand?" "Will A win the set in tennis against B?" "Is there a currant in this cubic foot of cake?" How enormously more pleasant and instructive it would be to deal with such multiple questions as "How much of this is likely to happen?" "What are the Probabilities that he will have 0, 1, 2, 3, or 4 aces, or that he will win all the various possible numbers of games in his set?" or "Draw a graph of the probabilities that the cubic foot of cake will contain all the possible numbers of currants from none at all up to as many as I want."

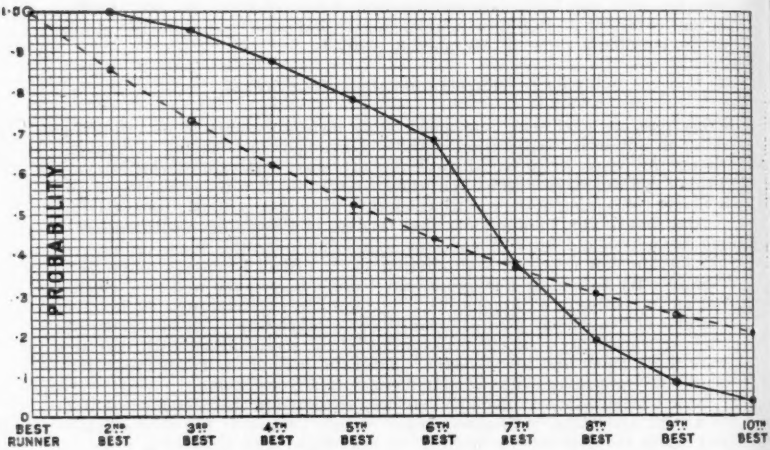
I believe that the difficulty of questions of this sort has been enormously overestimated; and that is why I have chosen a few examples, mostly very easy, of the kind of question which might bring home the ideas of Probability-Distribution and Frequency-Distribution, not only to professional mathematicians like you, but to the slightly-above-average specimens of the boys and girls whom you teach.

Graph 1.

I begin with running. Thirty-six runners enter for a race in which there are 6 prizes. Suppose they run in 6 heats of 6, the winner only of each heat starting in the final. The broken-line graph shows the various runners' chances of winning a prize. Even the 2nd best man is in considerable danger of failing to reach the final—and when you come down to the 6th best, who deserves the 6th prize, you notice that the odds are against his winning any prize at all. It will happen once in 5 times that your tenth best runner will win one of the 6 prizes. In fact this is a rotten system.

The continuous graph shows what happens if you run them in 4 heats of 9, and let the first two in each heat start in the final. Your second best man is safe for a prize now; all the best six have a much better chance on this system than the other, and the inferior runners, who deserve nothing, are likely to get what they deserve. The reason of this sharp drop between the

6th and 7th men is that if you are one of the best six you are sure to get a prize if you reach the final; while if you are 7th or worse you are still in danger of failing to win a prize even if you reach the final.



GRAPH 1.—Various runners' chances of winning a prize.

4 heats of 9: ————————

6 heats of 6: - - - - - - - - - - - -

I have a copy of this graph hung in my schoolroom, with the question written on it, "Which is the better system? Why?" The boys who notice it always answer, "The 6-heats one, because it is more like a smooth curve."

TABLE FOR GRAPH 2, showing the 6th best runner's chance of obtaining various places.

	4 heats of 9	6 heats of 6
2nd	—	·0001
3rd	·0012	·0240
4th	·0303	·1914
5th	·2306	·1996
6th	·4195	·0240
Fails in heat	·3185	·5610

In this graph * you are to imagine yourself the 6th best of these 36 runners. What you deserve is the 6th prize. Yes, but what are you going to get? The first column shows what happens if the race is run in 4 heats of 9,

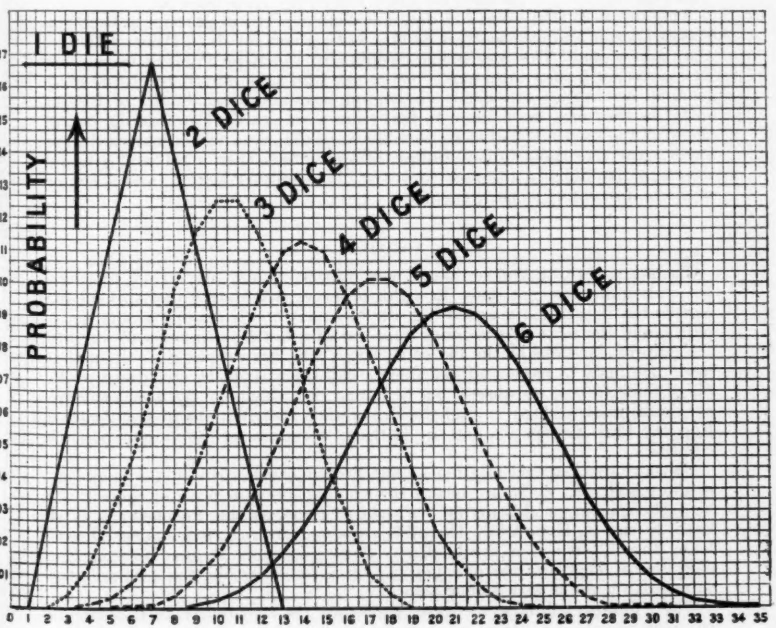
* Not printed here.

the first two in each heat running in the final. Here are your chances of being 3rd, 4th, 5th, or 6th, or failing to reach the final at all. One could wish the probability of failure less ; but at any rate, the thing you are most likely to get is what you deserve, 6th place ; so this 4-heat system is a decently fair one.

The second column shows what happens if there are 6 heats of 6, the winner only running in the final. You are the 6th best man. You may finish 3rd ; you are about equally likely to be 4th or 5th ; much the likeliest thing to happen to you is to be beaten in your heat and never reach the final at all, and the most unlikely of all your possible fates is to get what you deserve, 6th place.

I wish the iniquity of this system were more widely realised. When first I made these calculations I was so horrified that I took a large spade and went and widened the road where our School Mile starts, so as to make room for 9 starters, and I would urge on any of you who have any influence in your School Athletics to press for big heats rather than small, so as to ensure the maximum probability of the prizes going to the boys who deserve them. Even if you reject my general theory of a philosophy of life founded on Probability, allow the ghost of this dead theory to enlighten you in obtaining a better approximation to justice in School Sports than has been common in the past.

Graph 3.



Graph 3.—Probability-distribution of totals obtainable by throwing dice.

If you throw a single die (assuming it to be a fair one), the probability of scoring 1 is $\frac{1}{6}$. So is the probability of scoring 2, or 3, or 4, or 5, or 6. This is the Probability-Distribution—not a very stimulating graph.

With two dice the likeliest throw is 7, happening once in 6 times. The graph shows the probabilities of the various possible totals from 2 to 12, beginning and ending at the impossible totals 1 and 13, whose probabilities are 0.

With 3 dice we get a more interesting shape for the Probability-Distribution, and as the number of dice increases, the Probability-Distributions approximate closer to a curve which is the shape of an Admiral's hat if you draw it on a small vertical scale, or an ice-cream pudding if you draw it on a big vertical scale.

The construction of all the graphs I have shown you so far demands no calculation beyond Permutations and Combinations; in fact this one doesn't even get so far as that. It is easily within the range of the thoughtful-but-not-very-learned boy, for whom I believe we ought to cater more than we commonly do.

If you imagine a gun out to the right of the picture, so elevated that the Mean Point of Impact of its shells is at $10\frac{1}{2}$ on this scale, its hits will be distributed beyond and short of its Mean Point of Impact according to a Frequency-Distribution rather of this sort. In fact, if every shell lies just where it falls, and you go on firing millions of shells and letting them accumulate, they will form a hill the shape of an ice-cream pudding, with an equation of this type: $y=e^{-x^2\,2}$, a curve of which I shall say more later on.

This supposes that your gun does not change; but in practice a worn gun shoots shorter and also scatters its hits wider about its Mean Point of Impact than a new one. This means that people near the Mean Point of Impact are in less danger from a worn gun; but people who have moved 100 yards away from it, or who are crossing open ground, avoiding Mean Points of Impact so far as they can, have more to fear from worn guns. Towards the end of the War the Germans had some horribly worn old guns, which scattered their shells more like these broad stumpy admiral's hats than a tall and graceful ice-cream pudding—a disgrace to a first-class military nation, and a source of grave discomfort to their enemies.

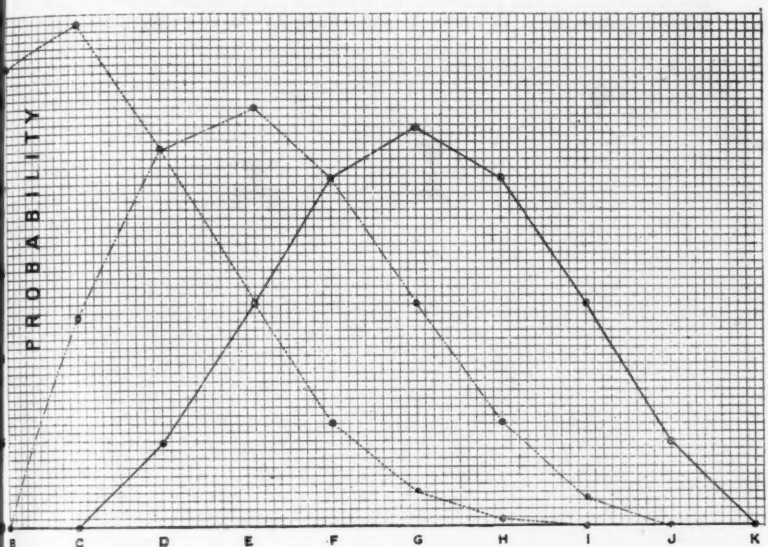
Graph 4.

The story of this graph is that a new master, supervising a row of seven boys whom he does not know by sight, observes that during the Anthem the middle boy is eating a poached egg, the boy next him is reading a pink newspaper, and the next is consuming rum out of a bottle. Anxious to identify the culprits, he looks at the list of the boys who are supposed to sit in that row. He finds 13 names printed there; call them *A*, *B*, *C*, etc., down to *M*. Six of these 13 boys are absent, but he has no idea which 6. With the evidence at our disposal we have to find the Probability-Distributions of the Poached Egg, the Pink Newspaper, and the Bottle of Rum among the various possible culprits.

Consider the Poached Egg first. The Probability-Distribution is obviously symmetrical. *K*, *L*, *M* are out of the running, because none of them could have three pious neighbours on our right of them, as the Poached-egg eater has. So are *A*, *B*, and *C*. The odds are about 20 to 1 each against *D* or *J* being the culprit; *G* is the likeliest. Notice a rough likeness between this continuous graph and the dice graphs. It is about as like the admiral's hat as one can make it with so few points.

The "flumpety-flumpety" graph represents the Probability-Distribution of the pink newspaper. It is an unsymmetrical variation on the admiral's hat, commonly found in Christmas crackers, while in the "pip-pip-pip" graph,

representing the Probability-Distribution of the bottle of rum, the asymmetry is exaggerated.



GRAPH 4.—Probability-distributions of
a Poached Egg: ————————
a Pink Newspaper: - - — - - — - - — - -
and a Bottle of Rum: ....

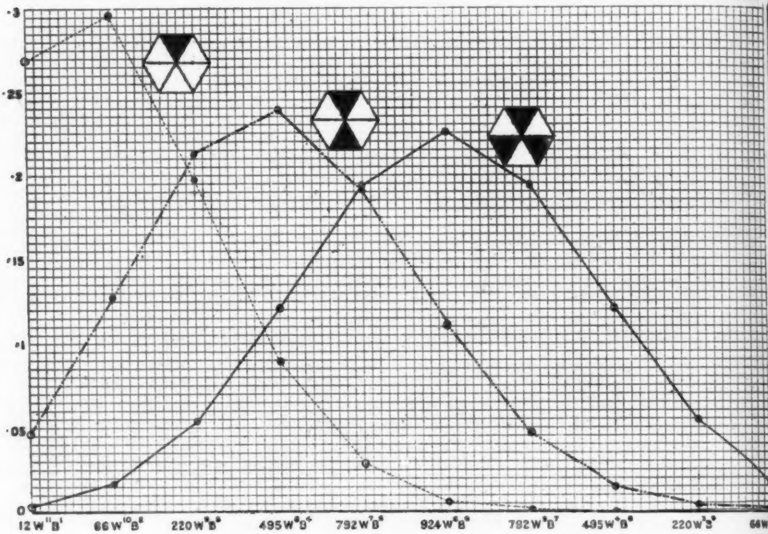
All these graphs required for their construction nothing beyond very mild Combinations, and are easily within the reach of thousands of boys and girls to whom the idea of Probability-Distribution is now entirely foreign.

GRAPH 5.

For comparison with these Probability-Distributions I have drawn graphs of the terms in the expansion of $(W+B)^{12}$. first taking W and B each equal to $\frac{1}{2}$ in order to get a symmetrical graph, next taking W as $\frac{2}{3}$ and B as $\frac{1}{3}$, which throws the maximum term earlier, then taking W as $\frac{5}{6}$ and B as $\frac{1}{6}$ to get this exceedingly unsymmetrical pattern.

As this expansion has 13 terms, not counting the zero terms at the ends, and the articles consumed in Graph 4 had only 7 possible claimants for each, it is impossible for the graphs to be exactly like each other. Actually no term of any of these Binomial Expansions coincides exactly with any probability indicated here; but the upper parts of these two families of graphs do agree to a very remarkable degree of approximation. In the Binomial graphs the interchange stations disappear; but there is very little else to give away the disagreement. It appeals to me as an astonishing thing that so many Probability-Distributions and Frequency-Distributions in Nature approximate so closely to the shape of these graphs of Binomial expansions. Some, of course,

actually are Binomial expansions. If you spin a top 12 times, and it is equally likely at every spin to come down with a white or a black side down, the symmetrical graph represents the Probability-Distribution of the various possible results, from 12 whites at one end to 12 blacks at the other, 6 of



GRAPH 5.—Graph of terms in the expansion of $(W+B)^{12}$,

when $W=B=\frac{1}{2}$: ————————

when $W=\frac{2}{3}$, $B=\frac{1}{3}$: — · · — · · — · · — · · —

and when $W=\frac{5}{6}$, $B=\frac{1}{6}$: ··························

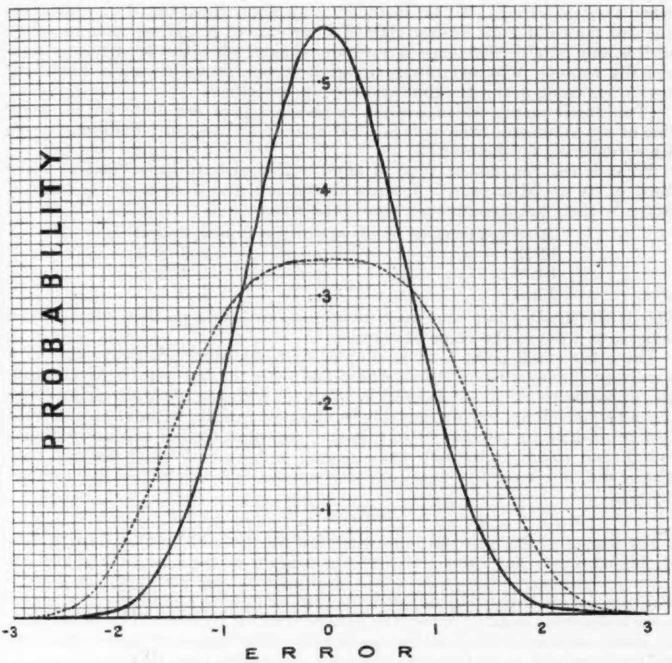
each being the most probable. The "flumpety-flumpety" graph gives the distribution for a top in which $\frac{2}{3}$ of the sides are white and $\frac{1}{3}$ black; 8 whites and 4 blacks is the most probable score this time. If the top has 5 white edges to one black, the probabilities of the various scores are the terms of this expansion when $W=\frac{5}{6}$ and $B=\frac{1}{6}$.

Perhaps I may suggest here that when boys learn the regular method of finding the maximum term of a Binomial expansion, it is worth while to let them draw some graphs of the terms like this, for three reasons. First, because it throws light on the problem actually before them, the finding of the maximum term, and on the reason for the method they use. Second, because it gives a splendid illustration of slide-rule work; the slide-rule is seen at its best in getting successive Binomial terms as they should be got, each term by multiplying the term before, and not by working out each term *de novo*. And finally, because these graphs pave the way for an appreciation of the ideas of Frequency-Distribution and Probability-Distribution.

GRAPH 6.

This is the first of the graphs I have shown you which needs a little Calculus for working it out.

Six quantities are known correct to the nearest whole number; they might, for instance, be the weights of six men, each measured correct to the nearest pound. Every one of them is subject to an error which is equally likely to be anything from $-\frac{1}{2}$ to $\frac{1}{2}$. Now add them up. The tall graph is the probability-distribution of the error in the sum, the likeliest error being nothing. When you mark the probability of an error of 1 as being ·22, what you really mean is that the chance of an error between 1 and $1+dx$ tends to the limit ·22 of dx if dx is small enough. This is the best ice-cream pudding I have shown you yet; it is indistinguishable by eye from the curve $y=e^{-x^2/2}$.



GRAPH 6.—Probability-distribution of errors in the sums
$a+b+c+d+e+f$: ————————
and $3a+b+c+d$: ··············

In the stumpy graph, instead of adding six independent quantities, you have added six, of which three are identical, as you do when you find the logarithm of V^3xyz. Notice how much likelier the bigger errors are to occur now, and how much less likely it is that your error will be between $\frac{1}{2}$ and $-\frac{1}{2}$, and your answer correct to the nearest whole number. Here we have the reason why, if you calculate V^3xyz by logarithms, you are so much less likely to get the last figure right than if you calculate $tuvxyz$. The chance that $3a$ is correct to the nearest whole number is evidently $\frac{1}{3}$. You might suppose that the addition of b, c and d to it would decrease this chance very considerably;

but actually the effect is only to decrease it in the ratio 191 to 192. In fact, if once you have got as far as finding the logarithm of V^3, you won't make your answer appreciably more likely to be wrong by throwing in the factors x, y and z as well.

Graphs of this type are too difficult for most boys to work out; but they might make good illustrations for teachers to use in explaining the principles on which the last figure of an answer got by Logarithms is sometimes trustworthy and sometimes not. All rules dealing with the reliability of the last figure are based on Probability; to admit this openly when we explain them would be honester, more enlightening to our pupils, and less disappointing when they go wrong. $3\frac{1}{7}$, for instance, is an approximation to π, which gets the 3rd figure of your answer right 5 times out of 6. If we make a secret of this, our pupils' confidence is naturally shaken when the 3rd figure is occasionally wrong. Whatever approximation we use, let us be candid about our reason for using it and the limitations of its usefulness.

[Two small points which I want to mention here are quite off the main line of my argument, such as it is; but as they fit in equally badly anywhere I am venturing on them now.

What is the commonest type of hand of 13 cards to have dealt to you? Most of us would answer, "4 cards of one suit and 3 each of the others." Actually there are 3 commoner types than this.

	21·55 per cent.	of hands	are	of the	type	4, 4, 3, 2
	15·52	,,	,,	,,	,,	5, 3, 3, 2
	10·58	,,	,,	,,	,,	5, 4, 2, 2
and	10·54	,,	,,	,,	,,	4, 3, 3, 3.

This is the kind of fact that may surprise a boy; and perhaps some of you believe, as I do, in the value of surprise.

The other small point is even simpler, but has some practical value. In a railway carriage 9 boy scouts quarrelled over the last remaining caramel, and appealed to me to act as umpire and allot it to one of them, giving every boy an equal chance. How can this be arranged quickly without an elaborate system of drawing papers out of a hat, or repeated tossing of coins with 1st round, 2nd round, ante-finals and drawing of byes? I won't tell you now what I did, but leave you to tell me afterwards. No, I didn't eat the caramel myself.]

Graph 7.

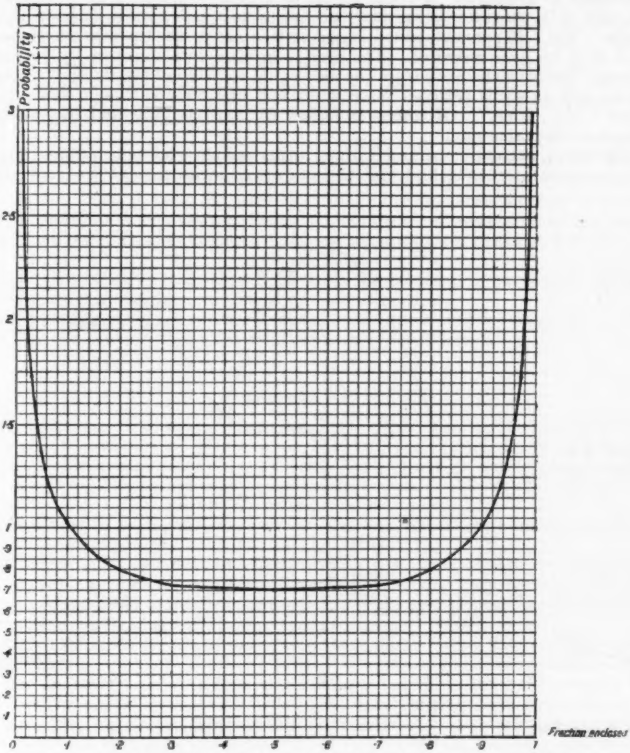
For fear of giving you the impression that all Probability-Distributions are of this admiral's hat shape, with the maximum in the middle, and coming down to nothing at the ends, I am showing you the best instance I know of a barge-shaped distribution, with the minimum in the middle. If a closed curve is drawn on squared paper, the squares which it crosses will have fractions of them enclosed. Here is the probability-distribution of those fractions. Big fractions and small fractions are much commoner than fractions about $\frac{1}{2}$. The keel of the barge is at $\frac{1}{2}$, and in that neighbourhood the probability is ·7071 of the mean; which you will recognise as sin 45°.

The calculations required for this graph are beyond the ordinary schoolboy; but when once he is familiar with the idea of Probability-distribution, it would be profitable to ask him why it is of this general shape.

Graph 8.

Here is the curve I have mentioned, the admiral's hat or ice-cream pudding, to which symmetrical Frequency-Distributions have such a wonderful way of approximating. It is sometimes called "The Normal Frequency Curve." I have put in the factor $\frac{1}{\sqrt{2\pi}}$ here so as to make its area come to 1. If you wipe out the 2 under the x^2, you get the same curve on a different horizontal

scale, but you upset the simplicity of the Radius of Gyration, much used in Statistical work, which is one unit of x if the equation is in this form. The point of inflexion is also at unit distance from the vertical axis.



GRAPH 7.—Probability-distribution of fractions of square enclosed by curve.

The Centres of Gravity of the two halves are of some interest. Their height is the maximum y of the curve divided by $\sqrt{8}$. Their distance from the axis would be twice the maximum y if the same scale were used for x and y.

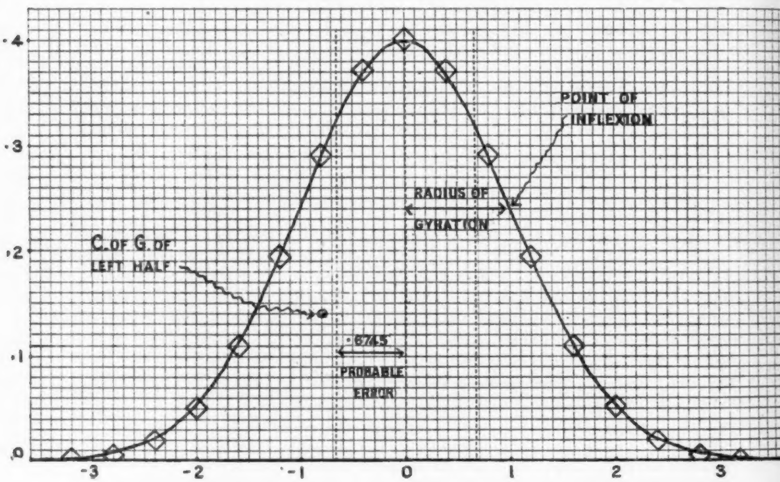
The calculation of the area is a beautiful piece of Integral, giving a good instance of three-dimensional Geometry coming to the rescue of a plane problem when plane Geometry is bankrupt. It can also illustrate the inter-relations of e and π at a stage of a boy's career where these have all the charm of novelty and mystery.

" Safe comes the ship to haven
  Through billows and through gales,
If once the Great Twin Brethren
  Sit shining on the sails."

A feeling which most of us have, I think, for e and π.

Now, how many of your better pupils who reach things like Moments of Inertia ever meet this beautiful curve? I had taken my degree three years before I ever heard of it, and then it was through the accident of my becoming interested in Eugenics. During a year as a Siege Gunner, I never heard a mention of it, though it is the curve according to which guns distribute their shells. The "50 per cent. zone," based on this curve, was the nearest relation to it that we ever heard of. The area is divided into quarters by vertical straight lines at a distance from the axis equal to ·6745 of the x-unit. A gun places half its shells between these limits and half outside them; hence the name "50% zone." Of course it differs for various guns and for various ranges of the same gun.

Surveyors, astronomers, and various other people who take approximate physical measurements, find that their errors of measurement are generally



GRAPHS 8 and 9.—$y=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, and terms in the expansion of $(p+q)^{24}$, when $p=q=\frac{1}{2}\sqrt[24]{2\cdot 5}=\cdot 51946$.

distributed in frequency according to this curve. They deduce, by a proof which I will spare you now, that when they have taken the mean of all their observations as their estimate of the true value of the measured quantity, the probabilities of the various other claimants to the title of "The true value" are distributed according to a curve of this sort. Therefore, so far as they can tell from the evidence at their disposal, the true value is equally likely to lie inside or outside these limits. They call this measurement the "Probable Error": and when you read that somebody has worked out the parallax of a star and got it to $1{\cdot}21'' \pm {\cdot}09''$ probable error, it means that he thinks the exact parallax equally likely "to be or not to be" between $1{\cdot}12''$ and $1{\cdot}30''$.

(The working by which he arrives at this conclusion is mostly beyond what your pupils can learn at school; but they ought at any rate to know what the astronomer's answer means when they read his parallax.)

And that is why you see the calculations of Probable Errors full of this factor ·6745. When I wrote to the late editor of Chambers' *Tables*, and suggested the inclusion of ·6745 and $\sqrt{2\pi}$ on his last page of *Numbers often used in Calculations*, in view of the growing importance of the Theory of Errors, he received my suggestion most cordially, and asked what ·6745 was all about. He had never heard of it before. I mention this, not to the discredit of him or his *Tables*, for which I have an enormous respect, but as an instance of the amazing ignorance of the use which practical men make of this curve, an ignorance prevailing apparently among some eminent mathematicians as well as among my own fellow-creatures.

To show you how closely Binomial terms approximate to this curve, I have drawn here a graph (9) of all the fairly big terms of $(p+q)^{24}$, taking this value of p and q so as to make the area of this graph equal to the area of the other. The square holes I have cut here have the Binomial terms as their centres, and are only ·002 of a square unit each: but you can see the curve through all of them, and in most cases crossing the square pretty near the middle.

The things I have been showing you so far are what I have been calling Probability-Distributions, all based on calculations.

The next few are based on experiment—I hope I shall be using the word in its orthodox sense if I call them Frequency-Distributions.

Graph 10.

I made this from a table I found in G. Udny Yule's *Introduction to the Theory of Statistics.* 8585 men were measured and classified into groups by inches. (The reason why these columns don't stand on graduations corresponding to whole numbers of inches needn't detain us now: it is connected with the system on which the measurements were taken.)

The height of each column represents the number of men who were of that particular stature.

Notice how closely the tops of these columns approximate to a Normal Frequency Curve, indicating that whether Nature is distributing shells of a gun, scores out of a dice-box, or men out of her own workshop, she chooses very much the same law of distribution for them.

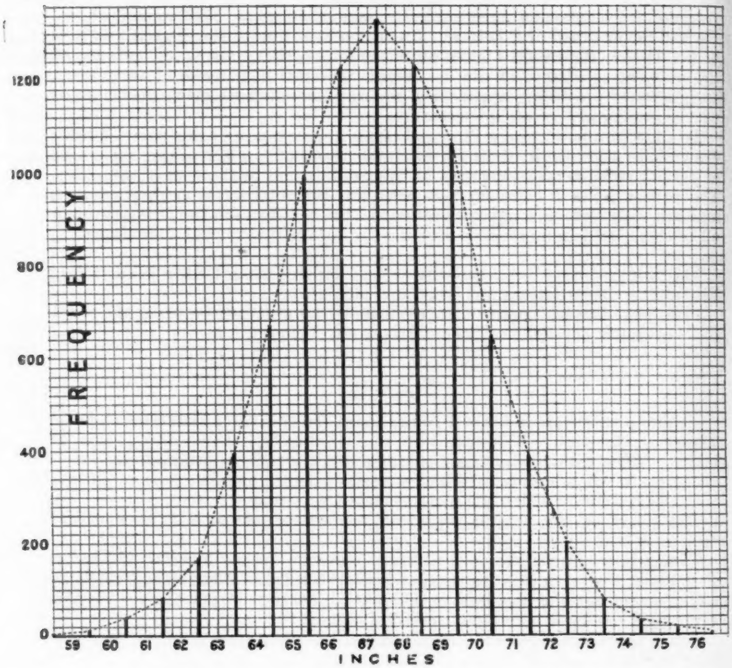
But if that is what Nature does about men's bodies, what does she do about their minds?

Here no direct numerical measurements are possible; but various approximate and empirical methods have been tried of estimating brain-power numerically. The remarkable thing about these different methods is that to such a large extent they give this same type of frequency-distribution: and when this happens repeatedly in estimating widely different qualities of mind by widely different methods, it points strongly to the conclusion that the real reason for this agreement is that the mental characteristics themselves actually are distributed according to the same law as physical characteristics such as stature.

If that is so, take this Graph 8, $y=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, as being the Frequency-Distribution of mathematical ability in this country. Here in the middle are the numerous average mathematicians, here on the left are the boys who don't see much difference between x^2 and $2x$, here at the extreme right are the Newtons; about here, say $x=2$, are most of us.

Now, if there is anything in my theory that these aspects of Probability illuminate life very greatly, it becomes an important question where the seed of them is to be sowed. From the fact that Wranglers leave Cambridge, and Doctors of Science publish Mathematical Tables, and Artillery Colonels direct the blazing off of thousands of tons of ammunition without ever having

heard of these things, I believe that an acquaintance with such ideas has so far been concentrated too far along this intelligence-scale.

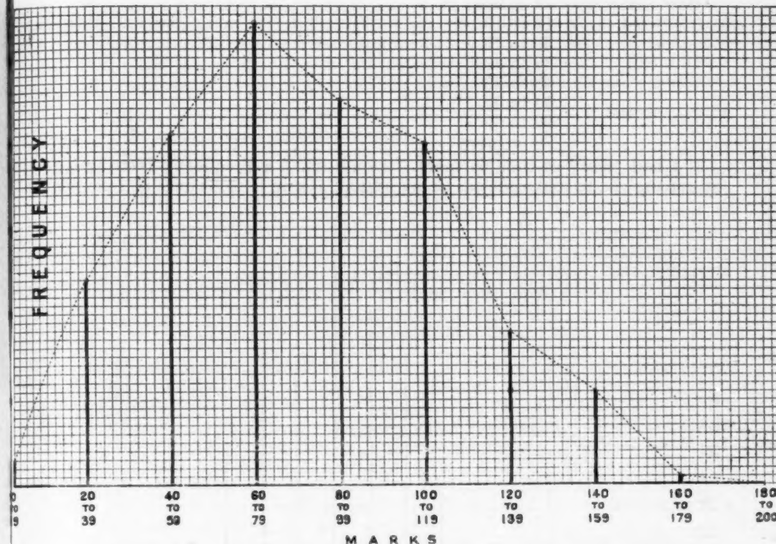


GRAPH 10.—Frequency-distribution of statures of 8585 men.

Now this (say $x=\frac{1}{2}$ to $x=1\frac{1}{2}$) is where the mass of the educable public opinion of the country lies: on one side of this region they do not take kindly to ideas—on the other side they suffer from shortage of numbers. If we believe, as I do, that these conceptions of orderly distribution arising from the unknown causes we call chance have light to throw on the problems "What is man?" and "What can we make of him?" then we ought to be pouring out liberally over this part of the rising generation as much of this Theory as it is capable of assimilating.

GRAPH 11.

236 boys are classified here into groups according to the numbers of marks they got. With smaller numbers it is to be expected that the approximation to a smooth-looking distribution should be rougher; but this is recognizably like the graph of the Binomial Terms when $W=\frac{2}{3}$ and $B=\frac{1}{3}$.



GRAPH 11.—Frequency-distribution of marks obtained for Mathematics by 236 boys in Remove at Eton, April 1919.

GRAPH 12.

The Frequency-Distribution of the height of the Barometer observed at Southampton on 4,748 days shows very much the same pattern, an unsymmetrical variation of the admiral's hat.

These data also are taken from Yule's *Introduction to the Theory of Statistics.*

For the next experiment you are to provide the data yourselves. Each of these packets contains 24 cards ; you are allowed one packet among four of you. Please take six cards each at random, not choosing them on any particular system. If you suspect the packets of 24 of being cooked, change a few cards with a neighbour who has had his from a different packet ; but please do it blindly and at random.

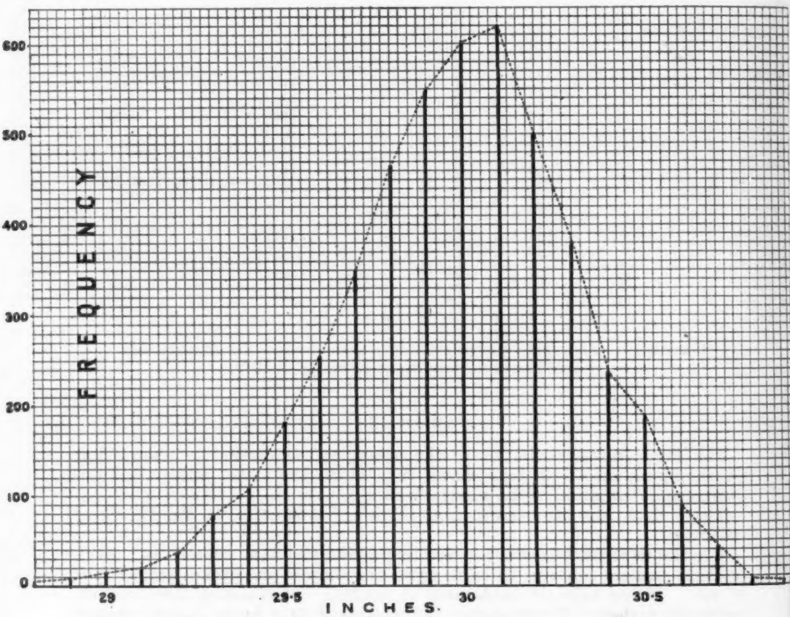
How many have 7 blacks
6 "
5 "
4 "
3 "
2 "
1 "
0 "
−1 " ?

[This experiment gave a good approximation to an "Admiral's hat" curve. The next was not tried, for want of time.]

Please add up your pips, counting Knave as 11, Queen as 12, King as 13.

How many have 9 or less
10 to 19
20 to 29
30 to 39
40 to 49
50 to 59
60 to 69
70 or more ?

If I had plenty of time to spare, I would make a Frequency-Distribution by getting you to make guesses at my weight, asking for multiples of 3 pounds



GRAPH 12.—Frequency-distribution of heights of the Barometer at Southampton.

as answers so as to avoid trouble with people who are more likely to guess 160 or 170 pounds than any number not ending with a 0.

MODEL.

The last thing I have to show you is an extension of this idea to 3 dimensions.

[The model which was shown is a hill of wood and putty ten inches high, with elliptical contours, the minor axis of each being about ⅘ of the major axis. Its vertical sections are all "Admiral's hat" curves of different heights and widths. It is mounted on a horizontal squared board graduated along the edges in inches, and is pivoted about a vertical axis, so that the axes of the elliptical contours can make any angle required with the edges of the board.]

Imagine a gun here, dropping its shells on to this place [the centre of the hill] as a mean point of impact. They are distributed beyond and short of this point according to this normal frequency-curve. Also they are distributed to right and left of it, but with less variation from the mean. The result is that if your shells lie where they fall they build a hill with elliptical contours. The equations of these contours are of this type: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, referred to the line of fire and its perpendicular as axes.

I have assumed so far that there is nothing about a long shot to make it likelier to go to the left or to the right than a short one. But now suppose that owing to some defect of the gun or the gunner long shots have a tendency to go more to the right and short ones to the left. Then here * is the shape of the hill your shells will build. The axes of the elliptical contours are no longer parallel and perpendicular to the line of fire. This is the fact that indicates a connection, or "correlation" between over-shooting and right deviation. The equations of the contours are something of this type now:

$$\frac{x^2}{a^2}-\frac{2Rxy}{ab}+\frac{y^2}{b^2}=1.$$

Notice that this xy term is the one that has turned the hill out of its old direction. It is the direct outcome of the cause that has made long shots and right deviation tend to occur together. That is why this R is called the "Coefficient of Correlation" between these two things. A glance at the equation will show you that if the contours are to be ellipses and not hyperbolas, R must be between 1 and -1. If R is 1 or -1, it means that y is a function of x, absolutely determined if x is known. The nearer R is to 1 or -1, the more is y governed, controlled and influenced by x; the nearer R is to 0 the more independent y is of x.

Now try man. If you choose pairs of grown men at random, taking them, for instance, in the order in which they pass the ticket-collector at a tube station, and represent every pair by a grain of sand laid on a board at a point whose x and y co-ordinates are the statures of those two men, you will build a hill rather like this but with circular contours.

But now if every pair consists of a man and his father, you get a surface like this * turned, and with elliptical contours. In fact, this surface was constructed so as to be the nearest possible surface of its type to the actual frequency-distribution of the statures of 1,078 men and their fathers, measured for Professor Karl Pearson's analysis of inheritance of stature. Its equation is $z=10e^{-\frac{1}{2}\left(\frac{x^2}{5\cdot47}-\frac{xy}{5\cdot43}+\frac{y^2}{5\cdot60}\right)}$, reckoned in inch units: its elliptical contours are got by taking this index as constant, and by comparison of these two forms $\left(\frac{x^2}{a^2}-\frac{2Rxy}{ab}+\frac{y^2}{b^2}\text{ and }\frac{x^2}{5\cdot47}-\frac{xy}{5\cdot43}+\frac{y^2}{5\cdot60}\right)$, R, the Coefficient of Correlation, is ·51; which you may interpret thus: "Inheritance from your father influences your stature rather more than half as much as if your stature depended on your father only." If the Correlation had been negative it would have meant that tall fathers have shorter sons than short fathers.

If you capture a large number of people and measure for each the number of aunts he possesses and the percentage of Calcium in his bones, I think you will find a high negative correlation between the two, much Calcium going with few aunts. I give you this, not so much as an instance of negative correlation as for the sake of warning you against the common error of supposing that, if two quantities are correlated, one "is caused by" the other. We often have strings of correlated figures given us for purposes of political

* Here the speaker turns the "hill."

propaganda, and are left to draw the inference, often false, that one thing is the effect of another. Aunts do not repel Calcium; nor does Calcium murder aunts. What happens is that as you grow older the percentage of Calcium in your bones increases, while aunts have a tendency to die in greater quantities than new ones are born. The two are correlated only because they both depend on age.

Now here we have a thing of vital importance—a way of expressing numerically that very vague and elusive thing, a "general tendency." We all know that there is a "general tendency" for tall parents to have tall children, and athletic or musical or mathematical parents to have athletic or musical or mathematical children; but here is a thing which lifts these general tendencies out of the realm of superstition, and puts them down for us in clean hard decimals. If the world we live in is moulded mainly by general tendencies, then one of the chief contributions which the mathematician should be making to the world is the spreading of clearer ideas as to how these tendencies may be quantitatively expressed and compared with each other.

I will make no secret of my own belief that of all these general tendencies none affects us more vitally than inheritance, not only of physical but also of mental, and possibly even moral, characteristics. This is not the right place for eugenic propaganda; but I do urge upon you as mathematicians to give your attention to the mathematical basis of the claims which we eugenists are making for the power of evolution, working through natural inheritance, to save the race or to ruin it—to understand the nature of the mathematical basis of our belief, in order that if we are wrong you may refute our dangerous errors, and if we are right you may throw in your lot with us.

But though the eugenic application appeals to me as the most vital, it is far from being the only one. The world is full of general tendencies, *laws out of focus*, broadly effective in the long run but never infallible in an individual case. "Honesty is the best policy." What does it mean? Certainly it doesn't mean that all schoolmasters are more prosperous than all brigands. If you enunciate it as an inflexible law you are refuted as soon as a thief grows rich. What is the real truth? That there is a positive correlation between honesty and prosperity, and that if you pile the population of the world on to a board, making the two co-ordinates of every person's position his honesty and his ultimate success, you are going to build them into a hill with elliptical contours whose major axes run into the corners of straight dealing meeting with its fair reward and swindlers languishing in gaol. That xy term is the one that does it: what you are for is to increase it. If you let that xy term disappear, then it will pay as well (in the narrower sense of the word) to be a bad man as a good one—and if you let it change its sign, then your minor axis is where your major axis ought to be, and back we go into the dark ages again.

But if you think this illustration too intangible, remember that mathematicians are investigating the correlation between such things as consumption, alcoholism, housing, condition of children's teeth and eyes, Indian monsoons and Nile floods, English weather and crops, and a variety of things which will appeal as concrete realities to those of you who regard my interest in honesty or posterity as the nightmare of a disordered brain.

I have no idea how much of this is new to you. I never heard a word of it at Cambridge. I have met many mathematicians there and since who know a lot about far more advanced subjects but are strangers to this. I should not be surprised to hear that it has had better treatment at London University than at Oxford or Cambridge. But we at school are absolutely debarred from teaching the most elementary aspects of it by the fact that if we do our boys will all fail in their examinations.

If you believe that this subject is valuable, and that the ordinary intelligent child will get a clearer idea of the world he lives in for knowing something of its elements, press for the inclusion of some of it in the commoner examina-

tions. I can't tell you how to do it; I know nothing of mathematical politics but I am convinced that if you really believe in it you will get it.

Boys, and perhaps girls too for all I know, are still leaving school putting their trust in superstitions like these:

"One thing is as good as another, because you never know what is going to happen, and it will all be the same in 100 years."

"What I do won't make much difference: there are lots of other people to dilute its effects."

"Gambling is a good way of making money."

"If you have been winning, your luck is in, and you will go on winning."

"If you have been winning, you will lose now, because the luck is sure to change."

"Parsons are always right."

"Parsons are always wrong."

"You'll never get on much if you try to be too good."

"Germans are wicked people."

"There always have been wars, and there always will be." *

For the slaves of all these superstitions there is only one remedy: "The truth shall make you free." And it is our business to consider whether a large part of the mathematician's contribution to that truth doesn't come under the head of Probability.

Solution of Problem on page 148.

Number the boys from 0 to 8. Ask one to produce a coin and tell you its date. Suppose it is 1904. Divide it by 9. The remainder is 5. Give the caramel to the boy numbered 5.

W. HOPE-JONES.

GLEANINGS FAR AND NEAR.

248. I have recently been *reading* (and it is curious that sometimes, when otherwise in mental activity, I seem to myself unable to read a page, or almost a sentence of German) more than a hundred pages of Grassmann's *Ausdehnungslehre,* with great admiration and interest. Previously I had only the most slight and general knowledge of the book, and thought that it would require me to learn to *smoke* in order to read it. If I could hope to be put in rivalship with Des Cartes on the one hand, and with Grassmann on the other, my scientific ambition would be fulfilled. But it is curious to see how narrowly, yet how completely, Grassmann failed to hit off the Quaternions.—W. R. Hamilton to A. De Morgan. [Graves' *Life*, iii. p. 441.] (Per Prof. Genese.)

249. I should like to hear about Grassmann—whom I am not likely to read. Between ourselves, I am disappointed with Germans—almost always. I have a new theory—take it. German intellect is an excellent thing, but when a German product is presented it must be analysed. Most probably it is a combination of intellect (I) and tobacco smoke (T). Certainly I_3T_1 and I_2T_1 occur; but I_1T_3 is more common, and I_2T_{15} and I_1T_{20} occur. In many cases metaphysics occurs (M); and I hold that $I_aT_bM_c$ never occurs without $b+c>2a$. —A. De Morgan to W. R. Hamilton. [*Loc. cit.* p. 446.] (Per Prof. Genese.)

250. One of Boswell's *affaires de cœur* was at Utrecht with a certain Mlle. Isabella de Zuylen, a sprightly and clever young woman, with a talent for conic sections and for composing romantic analyses of her own character.—*Blackwood's Magazine*, Nov. 1922, p. 633.

* Perhaps one may add the superstition that our present electoral system gives the country a Parliament which represents it.

THEOREMS ON FACTORIALS AND HOMOGENEOUS PRODUCTS DERIVED FROM A THEOREM OF LAGRANGE.

LEMMA. If p is a prime number, the sum of the products of the numbers 1, 2, 3, ..., $p-1$ taken r at a time, where r is any number less than $p-1$, is a multiple of p (*Lagrange*).

Let
$$f(x) \equiv (x+1)(x+2)\ldots(x+p-1)$$
$$\equiv x^{p-1} + A_1 . x^{p-2} + \ldots + A_{p-2} . x^{p-3} + A_{p-1}$$
for any integral value of p: then $A_1, A_2, \ldots, A_{p-1}$ are the sums in question.

Now
$$(x+p)f(x) = (x+1) . f(x+1),$$
i.e.
$$(x+p)[x^{p-1} + A_1 x^{p-2} + \ldots + A_{p-2} . x^{p-3} + A_{p-1}]$$
$$\equiv (x+1)^p + A_1(x+1)^{p-1} + \ldots + A_{p-2}(x+1)^2 + A_{p-1}(x+1).$$

Equating coefficients on each side, we have
$$p . A_1 + A_2 = C_2{}^p + C_1{}^{p-1} . A_1 + A_2,$$
$$p . A_2 + A_3 = C_3{}^p + C_2{}^{p-1} . A_1 + C_1{}^{p-2} . A_2 + A_3,$$
$$\ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p . A_{p-2} + A_{p-1} = C_{p-1}^{p} + C_{p-2}^{p-1} A_1 + \ldots + C_1{}^2 A_{p-2} + A_{p-1},$$
$$p . A_{p-1} = 1 + A_1 + A_2 + \ldots + A_{p-1}.$$

From which in turn, when p is a prime, since $C_1{}^{p-r}$ is not a multiple of p, $A_1, A_2, \ldots, A_{p-2}$ are multiples of p.

Cor. 1. Hence, from the last equation, since $A_{p-1} = \underline{|p-1}$, we have $1 + \underline{|p-1} =$ Mult. p (Wilson's Theorem).

Cor. 2. If x is any number prime to p, one of the numbers
$$x+1, x+2, \ldots, x+p-1$$
must be a multiple of p; hence $f(x)$ is always a multiple of p;
$$\therefore \; x^{p-1} + A_{p-1} \text{ is a multiple of } p.$$

Hence, by Cor. 1, $x^{p-1} - 1$ is a multiple of p (Fermat's Theorem).

THEOREM 1. If p is a prime greater than 3, then A_{p-2} is a multiple of p^2.

For if, in Lagrange's Theorem, we successively substitute for x the values p and $-2p$,
$$(p+1)(p+2)\ldots(2p-1) \equiv A_{p-1} + A_{p-2} . p + A_{p-3} . p^2 + \ldots + p^{p-1},$$
$$(2p-1)(2p-2)\ldots(p+1) = A_{p-1} - A_{p-2} . 2p + A_{p-3} . Ap^2 - \ldots + (2p)^{p-1};$$
$$\therefore \; 0 = A_{p-2} . 3p - A_{p-3} . 3p^2 + \text{a multiple of } p^4;$$
$$\therefore \; A_{p-2} = A_{p-3} . p + \text{a multiple of } p^3, \text{ if } p > 3,$$
i.e. A_{p-2} is a multiple of p^2, if $p>3$.

If, however, $p=3$, A_{p-3} is unity and not a multiple of p: hence, p must be greater than 3, in order that A_{p-2} should be a multiple of p^2.

Cor. If p is a prime >3, then
$$\underline{|p-1}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1}\right] \text{ is a multiple of } p^2.$$

For this expression is evidently A_{p-2}; this result was first stated by Mr. R. F. Davis in the *Educational Times*, I believe.

THEOREM II. If p is a prime greater than 3, and n is any whole number whatever, then

$$\frac{|\underline{np}}{(|\underline{p})^n} - |\underline{n} \text{ is a multiple of } p^3.*$$

(1) Let $n=2$; substitute p for x in the theorem of Lagrange, and we obtain

$$(p+1)\ldots(2p-1) = A_{p-1} + A_{p-2}\,.\,p + A_{p-3}\,.\,p^2 + \ldots + p^{p-1};$$

$$\therefore \quad \frac{|\underline{2p-1}}{|\underline{p}} - |\underline{p-1} \text{ is a multiple of } p^3;$$

$$\therefore \quad \frac{|\underline{2p-1}}{|\underline{p}\;|\underline{p-1}} - 1 \qquad \text{is a multiple of } p^3;$$

$$i.e. \quad \frac{|\underline{2p}}{(|\underline{p})^2} - 2 \qquad \text{is a multiple of } p^3.$$

(2) Let $n=3$; substitute $2p$ for x in the theorem of Lagrange, and, as in (1), we have

$$\frac{|\underline{3p-1}}{|\underline{2p}\;|\underline{p-1}} - 1 \text{ is a multiple of } p^3;$$

$$\therefore \quad \frac{|\underline{3p}}{|\underline{2p}\;|\underline{p}} - 3 \qquad \text{is a multiple of } p^3;$$

hence, from (1),

$$\frac{|\underline{3p}}{(|\underline{p})^3} - 2\,.\,3 \qquad \text{is a multiple of } p^3.$$

Continuing this process step by step, the theorem enunciated above is established.

Cor. If $n = a+b+c+\ldots$, it follows that

$$\frac{|\underline{np}}{|\underline{ap}\,.\,|\underline{bp}\,.\,|\underline{cp}\,.\ldots} - \frac{|\underline{n}}{|\underline{a}\,.\,|\underline{b}\,.\,|\underline{c}\ldots} \text{ is a multiple of } p^3.$$

THEOREM III. If p is a prime greater than 3,

$$2^{2(p-1)} - (-1)^{\frac{1}{2}(p-1)} \frac{|\underline{p-1}}{(|\underline{\tfrac{1}{2}(p-1)})^2} \text{ is a multiple of } p^3.$$

For if, in the theorem of Lagrange, we substitute $-\frac{1}{2}p$ for x and multiply by 2^{p-1}, we obtain

$$2^{p-1}|\underline{p-1} - (-1)^{\frac{1}{2}(p-1)}[1\,.\,3\,.\,5\ldots(p-2)]^2 \text{ is a multiple of } p^3;$$

$$\therefore \quad 2^{p-1}|\underline{p-1} - (-1)^{\frac{1}{2}(p-1)}\left[\frac{|\underline{p-1}}{2^{\frac{1}{2}(p-1)}\,|\underline{\tfrac{1}{2}(p-1)}}\right]^2 \text{ is a multiple of } p^3;$$

$$\therefore \quad 2^{2(p-1)} - (-1)^{\frac{1}{2}(p-1)} \frac{|\underline{p-1}}{[|\underline{\tfrac{1}{2}(p-1)}]^2} \qquad \text{is a multiple of } p^3.$$

THEOREM IV. If A_r is the general coefficient in Lagrange's theorem, A_r is a multiple of p^2 when and only when $r+2$ is a prime.

For, the values of A_1, A_2, ... can be calculated as polynomials in p, which are of constant form independent of the numerical value of p. But we have

* This is, I believe, the first proof to be given of this theorem, noted by me some years ago in the *Educational Times*.

shown that for a particular value of p, A_{p-2} is a multiple of p^2; hence the theorem follows.

Note. $A_r = H_r^{p-1}$, the sum of the homogeneous products of the numbers 1, 2, 3, ... $(p-1)$ taken r at a time. The equations in the Lemma hold for all values of p: hence we can find in turn

$$H_1^{p-1} = \frac{p(p-1)}{2},$$

$$H_2^{p-1} = \frac{p(p-1)(p-2)(3p-1)}{24},$$

$$H_3^{p-1} = \frac{p^2(p-1)^2(p-2)(p-3)}{48},$$

$$H_4^{p-1} = \frac{p(p-1)(p-2)(p-3)(p-4)}{5760}(15p^3 - 30p^2 + 5p + 2),$$

$$H_5^{p-1} = \frac{p^2(p-1)^2(p-2)(p-3)(p-4)(p-5)}{11520}[3p^2 - 7p - 2],$$

and so on. J. M. CHILD.

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251. I am glad to hear of your son's appointment. Anything in my power to fit him for success you may command. I have already done as much for him in the Elements of Geometry as I could—the rest he must do for himself. We now propose commencing with those of Algebra, and proceeding thro' that in like manner. With a similar course thro' Logarithms and Trigonometry he will be completely made up in the elementary Mathematics. This can all be accomplished, even with a knowledge of Oriental languages attaining in the meantime, before next Christmas. To be a *Mathematician*, however, depends, as I said before, on himself.

I shall be at home this evening, perhaps at five, but certainly at six, and shall be happy to see him then, as well as every Monday evening after. A weekly lecture will be sufficient for the purpose mentioned, with industry, on his part—an *hourly* one would be insufficient without it. Do not think from this reiteration that I hint inattention on his part—quite the reverse—he seems willing and deeply interested. But I know that young men who have not acquired Academic habits, are quite unaware of the necessary severity in order to attain the end in view. This, and a profound sense of the great utility of these sciences (in every profession but *my own* and in few more than his) makes me wish to impress him with a still stronger determination to proceed enthusiastically with them.

Pray give my kind respects to Mrs. Cunningham, and believe me, yours most sincerely, GEORGE DARLEY.—*George Darley*, Poet and Mathematician, to the Poet, *Allan Cunningham* (? 1836).

252. One wou'd think the Weakness of Man was intended by Nature to incite him to the Study of Mathematics. Other Animals She has sufficiently endow'd with Strength, Swiftness, and Offensive as well as Defensive Weapons, but has left Man altogether naked, and given him no other Portion but Wit and Invention: With these he encreases his Strength, acquires Swiftness, defends himself against all Attacks and from all Injuries, is bold enough for any Undertaking, raises himself to the Heavens, Studies and Measures their Motions, and applies 'em to his own use. Hence it is said, *That the Stars are made for his Service, and a Wise Man has a Right to Command them.*—Preface to vol. v. of the translation by J. T. Desaguliers of *Cursus Mathematicus: or, a Compleat Course of the Mathematicks*, 1712.

MATHEMATICS FOR EVENING TECHNICAL STUDENTS.

By Prof. H. T. H. Piaggio, D.Sc.

There is a large and well-defined group of students, in polytechnics and elsewhere, taking a course of mathematics which is very different from that given in schools. *Practical Mathematics*, as the subject is usually known, is regarded very unfavourably by many mathematicians, and no doubt it has many defects. However, it is possible that some of its critics have failed to realise the difficulties that occur in connection with evening classes. The present article is intended to show that Practical Mathematics meets a real need, and that it may be made really educational. No attempt will be made to conceal the grave weaknesses that often occur in the actual working of the classes. Some suggestions for improvement will be given, and it is hoped that those with experience of the work, either as teachers or inspectors, will give their views on the subject. Ultimately the Mathematical Association may be able to issue a report that will incorporate the most helpful suggestions received.

Let us consider the problem to be solved. A boy leaves an elementary school at fourteen, and secures employment in some branch of the engineering industry. After a time he presents himself at a technical college and says that he wishes to attend an evening class in Machine Drawing or in some other subject bearing directly upon his trade. He is persuaded to take up a systematic course, occupying two hours a night for three nights a week. In the first session the subjects studied are probably Machine Drawing, Applied Mechanics, and Practical Mathematics. This is by no means a light course to take after a hard day's work, and he would perhaps prefer to lighten it by omitting the mathematics, which does not seem to him to be really necessary. However, it is pointed out that the mathematics is needed to understand the other subjects.

If all goes well he continues his studies for several years, although there is always the possibility that he may lose heart or find the cinema or music-hall more attractive. (The classes generally close about Easter and do not open again until September, so there is no attempt to deprive him of his summer evening cricket or tennis.) The first duty of the mathematical teacher is to win the student's confidence and to give him practice in those calculations that occur frequently in the workshop. These are not very extensive. Perhaps a good knowledge of arithmetic (especially of decimals) and of mensuration is sufficient. Unfortunately, much of what was learned at the elementary school has been partly forgotten. The first year technical course commences by revising these subjects, and then deals with the use of logarithms, the numerical evaluation of engineering formulae, and the use of squared paper for interpolation, solving equations, representing formulae, determining maxima and minima, rates of growth, areas of indicator diagrams, etc. A little algebra and trigonometry are included, such as simple equations, right-angled triangles, and the use of trigonometrical tables. The course is rather disconnected and contains very little of the usual kind of mathematics, but it is only a beginning and probably the best one for technical students. This work is usually taken at evening continuation schools, and not at the technical colleges, which are now insisting on some previous knowledge as a condition of entrance.

The engineering subjects studied in later years involve much more advanced mathematics. A knowledge of differentiation and integration is required in connection with moments of inertia, bending moments, beam deflections, work of a variable force, connecting rod motion, and alternating currents. This differentiation and integration must be based upon a foundation of algebra and trigonometry. In fact, Practical Mathematics may be summed

up as Calculus (including a few special types of Differential Equations), with as much as possible of the academic mathematics required to lead up to this and with constant reference at every stage to engineering applications. Although the beginning of the course is very different from that of the usual one, the difference becomes less and less in the higher stages. It is found that the most satisfactory work is done in the final year. The weaker students never reach this class, and the stronger ones have been joined by a few who have had a secondary school education which enables them to dispense with the more elementary of the evening classes.

Some mathematicians may consider that the most glaring defect in the course outlined above is the complete omission of theoretical geometry. At the time when Prof. Perry commenced the activities that led to the setting up of Practical Mathematics classes all over the country, theoretical geometry meant Euclid. It was impossible for the evening student to master the numerous propositions contained in Euclid I.-IV. and VI., in addition to all his other mathematical and engineering studies. Moreover, the apparently obvious nature of the early theorems repelled him. The constructions were not such as he would ever use in his technical drawing. Some of the objections were removed in 1903, when Euclid ceased to be compulsory and most schools adopted other text-books. But the number of propositions was still rather large, though smaller than before. In the last few years some writers have reduced the number much further, including congruence, parallelism and similarity among the axioms and using freely ideas of symmetry. It is possible that in the future a short course on these lines may be evolved which may be acceptable to technical students. The writer would have to be both a sound mathematician and a skilled draughtsman with experience in a drawing office. Such men are rare, and when they exist they are generally too busy to spend their time catering for the needs of students. For the present there appears to be no time and very little inclination for theoretical geometry of the usual type.

Another criticism of Practical Mathematics, and a better-founded one, is that many text-books on the subject seem to be unnecessarily and aggressively inaccurate. It is certainly impossible in the time available to deal fully and accurately with some topics (such as the binomial and exponential series or the idea of a limit), but there is no reason why sound proofs should not be given when they are short and easy. When a sound proof is long or difficult it may be replaced by graphical considerations or analogies, but it should be honestly stated that these are not proofs. Unfortunately an evil tradition has grown up that lack of logic makes an argument peculiarly convincing to practical men. This may be due to historical reasons. The pioneers in Practical Mathematics were involved in much controversy, and they denounced the ordinary mathematical course with much vehemence. Much of what they said was more or less true and helped to bring about the reforms long advocated by the Mathematical Association. But the impetus of their attack on academic methods led them to reject the good as well as the bad, and their example has been followed by too many of their followers. This is an evil that teachers should try to eliminate.

In practice the chief difficulties arise not from the defects of the syllabus or the text-books, but from the somewhat casual way in which the students join the classes. The college authorities make great efforts to arrange everything on the opening night, but many students do not attend until two or three weeks later. Two friends generally try to join the same class, although this may be too advanced for one of them. Those who have deferred the commencement of their studies until several years later than usual, are often reluctant to take the urgently needed preliminary course in company with boys fresh from school. Then there is always a falling off in the attendance after the Christmas holidays. In the past many students have lacked sufficient incentive to keep steadily to the work. However, an arrangement has

just come into force that may do something to remedy this. National Certificates, endorsed by the Institution of Mechanical Engineers, will be awarded to those who comply with certain conditions regarding attendance, homework, and examinations. Two such certificates will be issued, one obtainable after three years and the other after two more years.

To conclude, it is claimed that Practical Mathematics is an indispensable part of evening technical work, and that it attracts evening students in a way that the ordinary type of mathematics never did. It has grave defects, but these are not inherent in it, and they can be removed if sufficient effort is made. When this is done, the subject will form a logical whole, different from an ordinary course occupying the same time, but equally worthy of respect. However, these may be the views of only one individual. The General Teaching Committee of the M.A. is considering whether it would be desirable to issue a report on this subject, and the chief reason for this article is to elicit the views of others interested, who are requested to communicate with the Hon. Sec. of the General Teaching Committee (Mr. R. M. Wright, Eton College, Windsor).

H. T. H. PIAGGIO.

University College, Nottingham.

Appendix. By the courtesy of the East Midland Educational Union a few typical questions from their examination papers in Practical Mathematics are given below:

(1) A cylindrical chimney-stack 50 feet high and of mean diameter 2 feet is made of steel plate, of average thickness $\frac{3}{8}$ inch thick. Neglecting the overlap of the plates and rivets, find the approximate weight of the chimney. [1 cub. in. steel weighs 0·28 lbs.]

(2) The following quantities x and y are supposed to be connected by the law $y = ax^n$:

x	2	3	4	5
y	16·8	56·7	134	263

By plotting $\log y$ against $\log x$, find whether this is so, and find the values of n and a.

(3) The power given by a certain current sent through an external resistance r is given by the formula

$$P = \frac{100r}{(r+5)^2}.$$

Find r and P when P is a maximum.

[*Note.* If y is a maximum, $\frac{1}{y}$ is a minimum.]

(4) If a gas is subject to the law $pv^\gamma = K$,

where p = pressure in lbs. per square inch,

v = volume in cubic inches,

$\gamma = 1{\cdot}41$,

find the work done as 1000 cubic inches at atmospheric pressure ($p = 14{\cdot}75$) is compressed to 600 cubic inches.

[*Note.* Work $= -\int p\,dv$.]

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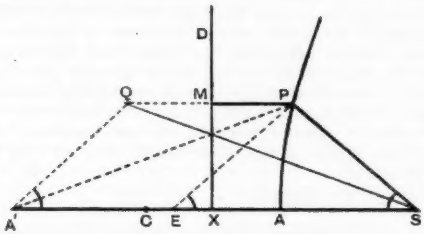
253. Algebra—that paradise of the mind, where it may enjoy the fruits of all its former labours, without the fatigue of thinking.—Quoted by De Morgan in Preface to his "Probability," from p. xii of *A Quaint Editor of Euclid.*

MATHEMATICAL NOTES.

717. [K¹. 21. b.] *Construction for Points on Newton's Trisecting Hyperbola* ($e=2$).

With the usual notation, the directrix DX is the perpr. bisector of SA'.

Take any pt. E on CS and with centres E, S and radii each equal to EA' describe circles cutting in P; then P is a pt. on the curve.



Produce the perpr. PM on DX to Q making $MQ=PM$.

Then $SPQA'$ is symmetrical about DX.

$$\therefore\ Q\hat{A}'S = P\hat{S}E = P\hat{E}S.$$

$$\therefore\ EPQA' \text{ is a parallelogram and } EA' = PQ = 2PM.$$

J. G. Furtado.

718. [V. 1. a. ζ] *The Teaching of Parallels to Boys of about* 12.

The starting-point in learning the idea of parallelism is not necessarily the one now conventional, and the conventional treatment leads to the impasse familiar to all teachers of geometry where the aid of Playfair's Axiom has to be invoked. The boy of 12 cannot follow lengthy philosophical arguments in the matter, and he in most cases accepts the statement made by his teacher and his book, in other cases he is sceptical. Postponement of the reproduction of the proofs for a few years will help some boys; probably most boys will not grasp the argument even then. The cause of this is the impossibility to visualize the infinite distance implied in the words "never meet" or their equivalents.

The following treatment is suggested; it has been found that comparative beginners readily grasp the ideas and easily reproduce the argument.

Definition. The angle between two straight lines is the amount of turning each line must undergo to occupy the position of the other.

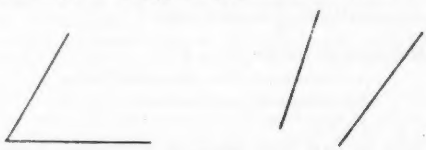


Fig. 1.

Definition. When the angle between two straight lines is zero, the lines are said to be parallel.

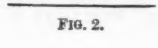


Fig. 2.

If two straight lines are drawn from a point, and the angle between them is zero, they coincide with each other.

FIG. 3.

Hence, two parallel lines do not meet each other when produced, for if they met they would have to be coinciding with each other.

FIG. 4.

Theorem. If a straight line cuts two parallel lines, the exterior angle is equal to the interior opposite angle on the same side of the cutting line.

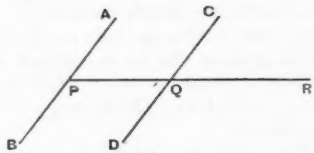


FIG. 5.

Let the straight lines AB, CD be parallel, and let the straight line PQR cut them.

It is required to prove that the $\angle RQC =$ the $\angle RPA$.

Proof. The difference of the angles RQC, RPA is the amount of turning between the straight lines QC, PA. But this is zero, by hypothesis.

$$\therefore \text{ the } \angle RQC = \text{the } \angle RPA. \qquad \text{Q.E.D.}$$

In the converse theorem, given the $\angle RQC =$ the $\angle RPA$, to prove the lines parallel, using the same figure :

The difference of the angles RQC, RPA is the amount of turning between the straight lines QC, PA. But the difference is zero, by hypothesis.

Hence the angle between DC and BA is zero, *i.e.* DC, BA are parallel.

Formal reproduction of these proofs by boys can be postponed; this is, however, hardly necessary.

The above treatment is analogous to the treatment of perpendicular lines; lines are perpendicular when the angle between them is a right angle (defined either in the usual way or as a quarter of a complete turn); they are parallel when the angle between them is zero.

It introduces the dynamical idea of rotation, which is a distinctly useful one. Also many children are naturally attracted by an object in motion in preference to the stationary.

Liverpool Institute. H. A. BAXTER.

719. [X. 10. b.] *On a Chess-board Problem.*

Mr. H. E. Dudeney gave in the February number of the *Lyons Mail* a neat arithmetical method for finding the number of different ways in which a king may move from the left bottom square of a chess-board to any specified square, under the condition that he always gets nearer to it. An algebraic treatment is not without interest as an extension of an old problem about a town with streets parallel to the sides of a rectangle, the novelty being that a king may move diagonally as well as along rows or columns, but always to a next adjacent square.

The position of any square may be given by m, the number of moves along the bottom row to reach the file of the square, and n the number of moves up the file to reach the square.

If one path of the king consists of p moves along rows, q along files, and r along diagonals, it may be permuted in

$$\frac{(p+q+r)!}{p!\,q!\,r!} \text{ ways.}$$

Now a path to (m, n) must consist of either

m moves along rows and n along files, or,

1 diagonal move, $(m-1)$ along rows $\times$ $(n-1)$ on files, or,

2 „ moves, $(m-2)$ „ „ $(n-2)$ „ or,

etc. The number of ways required is therefore

$$\frac{(m+n)!}{m!\,n!}+\frac{(m+n-1)!}{(m-1)!\,(n-1)!\,1!}+\frac{(m+n-2)!}{(m-2)!\,(n-2)!\,2!}+\text{etc.}$$

Using m_r to denote ${}_mC_r$, we may rewrite the series

$$(m+n)_n+(m+n-1)_n \,.\, n_1+(m+n-2)_n n_2+\text{etc.} \quad \text{.................(1)}$$

Another form may be given to this by the method of Finite Differences. Let $E=1+\Delta$ be the operator turning m into $m+1$, then (1) may be written

$$(E+1)^n m_n=(2+\Delta)^n m_n;$$

then, since $\Delta m_r=m_{r-1}$, we get

$$2^n m_n+n_1 2^{n-1} m_{n-1}+n_2 2^{n-2} m_{n-2}+\text{etc.} \quad \text{..........................(2)}$$

And this series $=$ coefficient of x^n in $(2+x)^n(1+x)^m$.

When the specified square is on the diagonal through (0, 0), $m=n$, and we require the coefficient of x^n in $(x^2+3x+2)^n$, or,

$$\text{the constant in } \left(3+x+\frac{2}{x}\right)^n=3^n+3^{n-2}n_2 2_1 2+3^{n-4}n_4 \,.\, 4_2 2^2+\text{etc.,} \quad \text{......(3)}$$

an odd power of $x+\dfrac{2}{x}$ not containing a constant term.

Putting $m=n=7$, the number of different ways to the opposite corner of the board is found to be 48,639.

Mr. Dudeney's method is based on the observation that any square can only be reached by passing through one of the three adjacent squares. If, then, $f(m, n)$ be the required number of ways, we must have

$$f(m+1, n+1)=f(m, n)+f(m+1, n)+f(m, n+1),$$

or, if E' be the operator turning n into $n+1$,

$$(EE'-1-E-E')f(m, n)=0,$$

i.e. $$\{(E-1)(E'-1)-2\}(fm, n)=0,$$

or $$(\Delta\Delta'-2)f(m, n)=0.$$

There is no difficulty in proving that the above series (1) and (2) satisfy this equation. R. W. GENESE.

720. [V. 9.] The following characteristic note by De Morgan on the effects in his day of "coaching" for the Cambridge Tripos—a method of teaching which he regarded as detrimental to originality—may interest some readers.

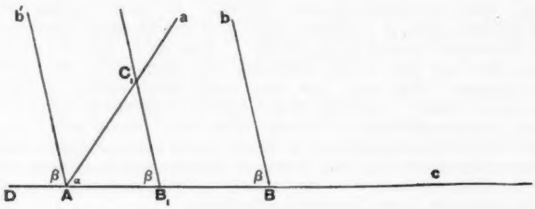
"I had," writes De Morgan, "a very early acquaintance with these Lagrangian views. In 1826 (long vacation), when I ought to have been cramming, I was eviscerating the *Mécanique Analytique*, and got out a full perception of the meaning of this magnificent generalization of Lagrange. Not having a private tutor—a disadvantage of which I feel the benefit every day of my life—I was able to carry it on without much opposition. I am afraid assimilators and digestors as such, are not valued as much as the foragers and feeders." W. W. ROUSE BALL.

721. [V. 1. a. ζ] *The Postulate of Parallels.*

Bonola's description* of Wallis's application of his postulate "that to every figure there exists a similar figure of arbitrary magnitude" to prove Euclid's postulate of parallels.

It has occurred to me that, as Bonola's book, to which reference has been made on p. 410, *Vol.* xi., may not be in the hands of many of the readers of the *Gazette*, it would be well to give the substance of Bonola's account.†

Let a, b be two straight lines intersected at A, B by the transversal c. Let α, β be the interior angles on the same side of c, such that $\alpha+\beta$ is less than two right angles.



Through A draw b' so that b and b' form with c equal corresponding angles. Then, since $\alpha+\beta$ is less than two right angles, it follows that b' will lie in the angle aAD adjacent to aAc.

Let the line b be now moved continuously along the segment BA so that the angle which it makes with c remains always equal to β until it reaches b'.

Before it reaches its final position b' it must necessarily‡ intersect a. Suppose that when b occupies the position B_1C_1 it intersects a at C_1 and c at B_1. In this way a triangle AB_1C_1 is determined, with the angles at A and B_1 equal to α and β respectively.

But by Wallis's postulate of the existence of similar figures, upon AB taken as the side corresponding to AB_1 it must be possible to construct a triangle similar to AB_1C_1, which may be called ABC, C being the third vertex but not shown in the figure.

This is equivalent to saying that the straight lines a, b must meet in a point, namely the third angular point C of the triangle ABC. Hence if $\alpha+\beta$ be less than two right angles, then a and b meet.

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* Bonola's *Non-Euclidean Geometry*, p. 16.

† It should be understood that Wallis's argument amounts to an endeavour to substitute one postulate (the postulate of similarity) for another (the postulate of parallels). Although Bonola does not criticise Wallis's argument, it will be seen from what follows that that argument has not been generally accepted. The first valid proof of the equivalance of the two postulates is (I believe) that given by Saccheri in his *Euclides ab omni naevo vindicatus*, but this is of very great length; the second is that given by Professor Nunn (see p. 412); and the third is that on pp. 410-413 above. It seems to me that one of the two last named proofs, or some equivalent proof, should find a place in every text-book of Elementary Geometry in place of Euclid's Postulate of Parallels. The proof given by Clifford (*Common Sense of the Exact Sciences*, pp. 71-2) is, I believe, open to objection.

‡ At this point another writer, M. Cantor (*Geschichte der Mathematik*, Vol. 3, p. 26), enters a caveat in his account of Wallis's work, but does not explain the grounds of his uneasiness in respect of the argument up to this stage. It seems to me that Wallis here makes a further appeal to intuition (in addition to that contained in his postulate), which amounts to assuming that if the intercept made by the lines a and b on the line c is small enough, and if $\alpha+\beta$ is less than two right angles, then a and b will meet on that side of c on which the angles α and β are situated. This looks to me very much like a *petitio principii*, and is certainly quite unnecessary to obtain the required result, as has been shown on pp. 410-413 above.

M. J. M. Hill.

722. [V. 9.] *Reminiscences of Sylvester.*

"Rubicund, burly, of commonplace exterior, Professor Sylvester was as full of whimsicalities and contradictions as it is possible for any human being to be. Of his astounding, his unrivalled mathematical capacities and achievements he took small account. To perfect his 'upper C,' for he greatly prided himself on his vocal accomplishments, lightly and elegantly to jump over a stile, and to translate an ode of Horace in accordance with his own laws of syzygy, these were the ambitions of the greatest expert in modern algebra. 'Now for my upper C,' he would say at the house of his old friend Madame Bodichon, that lady delighting to humour him. So one of the party sat down to the piano, and again and again the Professor repeated his upper C.

"During the summer we used to meet at Madame Bodichon's country house in Sussex. There happened the stile incident. We were crossing some fields when his hostess, then in brilliant health and spirits, very dexterously took a stile or five-barred gate, I forget which. 'Dear me!' said the disconcerted Professor, who had just before managed the business in the slowest and clumsiest fashion, 'dear me! you must really teach me how to get over a stile, you really must.' And the lesson was good-naturedly given, when the first living mathematician in Europe, who could easily solve algebraical problems, the very contemplation of which would make ordinary brains reel, very nearly dashed out his own brains in the attempt to clear a stile. His two lady companions rushed to the rescue, or without doubt he would have fallen head foremost, doing himself deadly harm.

"It was the Professor's habit, no contemptible one, to carry a little notebook about with him, and therein to jot down any remark that appeared suggestive or original. Some of these jottings, pencilled when I was by, are alluded to in his 'Laws of Verse.' When translating Horace, with Boileau the Professor could say, *Je cherche et je sue*, but he made his friends seek and sweat too.

"I can see him now, standing on the hearth-rug of my tiny drawing-room, reciting his version of the famous ode to Maecenas. An excellent version it is, but few readers would guess the cost to its author in time and labour. In the original translation the first line had run thus:

Tyrrhenian progeny of kings,

finally altered as follows:

Birth of Tyrrhenian regal line!

In that happy amendment I claim some share. Again and again the Professor recited his ode on the hearth-rug, and again and again we argued about the word 'progeny.' It sounded in my ears so very unpoetical, so very unmusical. At last he gave way, and certainly 'regal line' is a vast improvement.

"Whimsicalities apart, Sylvester was a great and estimable man, and let us not forget the fact, in one sense victim of nineteenth century intolerance. He was a Jew, and, in spite of his brilliant mathematical reputation at Cambridge, could neither hope for the much-coveted Smith's Prize or Fellowship and Professorship in his University. These good things, forsooth! were reserved for adherents of the Thirty-nine Articles and members of the Established Church.

"The greatest mathematical genius of his generation was reduced to the drudgery of teaching, and had to content himself with Transatlantic honours. Sylvester, it is said, deeply felt the injustice, and well he might. 'Unhappily,' he wrote in those days to a young Nonconformist mathematician of great promise, 'there are very few positions in this university offering a suitable nest for the fostering of scientific progress of an abstract kind. All such berths are appropriated by the universities, which are positive evils and impediments to all born out of the pale, or at least to all who do not flock

within the pale of the Established Church; their existence precludes the State encouragement which would otherwise be bestowed indiscriminately on all.'

"When aged seventy-six, he was indeed named Savilian Professor to the University of Oxford, and Fellow of New College."—*Reminiscences* by M. BETHAM EDWARDS, p. 124.

723. [C. 1. f.] *The "doubtful case" in Maxima and Minima.*

In considering the stationary values of a function of two independent variables $f(x, y)$, text-books on the Calculus usually omit altogether the "doubtful case," when $f_x = f_y = 0$ and $f_{xy}{}^2 = f_{xx}f_{yy}$, as being too difficult to include in an elementary treatment of the subject.

This case, however, may be treated in a general, but quite elementary way by the use of infinitesimals. A similar method may be followed in discussing the nature of a singular point on an algebraic curve.

The method is here outlined.

If the point (x, y) gives a maximum or a minimum of $f(x, y)$, then

$$\Delta f(x, y) \equiv f(x+h,\ y+k) - f(x, y)$$

must not change sign for all values of h, k which are sufficiently small; and provided they are small enough, the values of h, k need not necessarily be of the same order of smallness.

$$\Delta f(x, y) = hf_x + kf_y + \tfrac{1}{2}(h^2 f_{xx} + 2\,hkf_{xy} + k^2 f_{yy}) + f_3(h, k),$$

where f_3 is of the third degree in h, k.

A necessary condition is found to be, by the usual argument, that $f_x = f_y = 0$.

Hence at a stationary point

$$\Delta f = \tfrac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) + f_3(h, k).$$

Suppose k is of order of smallness 1, and compare the order of h with that of k.

There are three cases:

(i) If the order of $h < 1$, the leading term is $\tfrac{1}{2}h^2 f_{xx}$ (provided $f_{xx} \neq 0$), which does not change sign.

(ii) If the order of $h = 1$, the leading terms are of order 2,

$$\tfrac{1}{2}(h^2 f_{xx} + 2\,hkf_{xy} + k^2 f_{yy}),$$

which changes sign if $f_{xy}{}^2 > f_{xx}f_{yy}$, and does not if $f_{xy}{}^2 < f_{xx}f_{yy}$.

(iii) If the order of $h > 1$, the leading term is $\tfrac{1}{2}k^2 f_{yy}$, which does not change sign.

(It will be supposed throughout that the terms of degree 2 do not all vanish, as if this is the case the conditions for maxima or minima are generally more complicated.)

It is easy to see that if $f_{xy}{}^2 < f_{xx}f_{yy}$, the sign of the leading terms in case (ii) is the sign of $\tfrac{1}{2}h^2 f_{xx}$ or $\tfrac{1}{2}k^2 f_{yy}$.

Hence, if $f_{xy}{}^2 > f_{xx}f_{yy}$, there is no maximum or minimum.

If $f_{xy}{}^2 < f_{xx}f_{yy}$, there is a maximum or a minimum.

In the "doubtful case" $f_{xy}{}^2 = f_{xx}f_{yy}$, cases (i), (iii) remain unaltered.

In this case,

$$\Delta f = \frac{1}{2f_{xx}}(f_{xx}h + f_{xy}k)^2 + \tfrac{1}{6}(h^3 f_{xxx} + \ldots + k^3 f_{yyy}) + f_4(h, k).$$

When h, k are of the same order, the leading terms do not change sign except when $f_{xx}h + f_{xy}k = 0$ (to the first order).

When $f_{xx}h + f_{xy}k = 0$, substituting for h, we have in general,

$$\Delta f = Ak^3 + Bk^4 + \ldots,$$

which changes sign with k, unless $A = 0$.

Hence it is evidently necessary that $A = 0$, *i.e.* that $f_{xx}h + f_{xy}k$ be a factor of the terms of third degree.

This is not in general sufficient, as by putting

$$h=-\frac{f_{xy}}{f_{xx}}k+Ck^{\frac{3}{2}},$$

we may obtain a leading term of odd degree.

The further consideration of this case is simplified by putting

$$h+\frac{f_{xy}}{f_{xx}}k=H, \quad i.e.\ h=H-\frac{f_{xy}}{f_{xx}}k.$$

We obtain for Δf an expression of the form

$$a_2H^2+(a_3H^3+b_3H^2k+c_3Hk^2+d_3k^3)+(a_4H^4+\ldots+e_4k^4)+\ldots.$$

If H is of order $< \frac{3}{2}$, the leading term is a_2H^2, which does not change sign.

If H is of order $> \frac{3}{2}$, the leading term is d_3k^3, which changes sign.

Hence it is necessary that $d_3=0$.

When this is so, we have

$$\Delta f=a_2H^2+(a_3H^3+b_3H^2k+c_3Hk^2)+(a_4H^4+\ldots+e_4k^4)+\ldots.$$

If H is of order <2, the leading term is a_2H^2, which does not change sign.

If H is of order 2, the leading terms are of order 4, $a_2H^2+b_3Hk^2+e_4k^4$, which change sign if $b_3^2>4a_2e_4$, and do not change if $b_3^2<4a_2e_4$.

If H is of order >2, the leading term is e_4k^4, which does not change sign.

Hence there is no maximum or minimum if $b_3^2>4a_2e_4$.

There is a maximum or minimum if $b_3^2<4a_2e_4$.

In the doubtful case, when $b_3^2=4a_2e_4$, the expression may be written

$$\Delta f=a_2\left(H+\frac{b_3}{2a_2}k^2\right)^2+a_3H^3+\ldots.$$

Again, put $H+\frac{b_3}{2a_2}k^2=L$ or $H=L-\frac{b_3}{2a_2}k^2$, and the expression becomes

$$\Delta f=a_2L^2+(a_3'L^3+b_3'L^2k+c_3'Lk^2+d_3'k^3)+\ldots,$$

which may be discussed as before.

It may be proved that this process must terminate.

In practice, when such an expression as

$$a_2h^2+(a_3h^3+b_3h^2k+c_3hk^2+d_3k^3)+(a_4h^4+\ldots+e_4k^4)+\ldots,$$

in which the coefficients are given as numbers, is considered, it is a good plan to strike out terms which are of higher order than terms already written down, whatever the order of h, k.

Thus, provided none of the above coefficients is zero, a_3h^3, b_3h^2k are both of higher order than a_2h^2. e_4k^4 is of higher order than d_3k^3, whereas c_3hk^2, d_3k^3 are both doubtful.

Then a_2h^2 and the remaining term which contains the highest power of k may be equated to zero to give an order for h, and the orders of the different terms may be written underneath them.

Aug. 31, 1923. H. V. MALLISON.

724. [L¹. 1.] *Some Notes on Projective Geometry.*

I. In elementary courses on Projective Geometry, based on the metrical definition of Cross Ratio, projections are often used without proof of the possibility of effecting them.

The following line of argument fills the gap.

(1) If two ranges are equicross they are projective.

(2) If two pencils are equicross they are projective.

For if transversals cut the pencils (vertices O_1 and O_2) at $(A_1B_1C_1\ldots)$, $(A_2B_2C_2\ldots)$ respectively, $(A_2B_2C_2\ldots)$ can be projected into $(A_1B_1C_1\ldots)$.

If O_3 is the projection of O_2, either O_3 coincides with O_1 or the figures $O_3(A_1B_1C_1\ldots)$ and $O_1(A_1B_1C_1\ldots)$ are coaxial, and therefore in perspective.

Hence $O_1(A_1B_1C_1\ldots)$ and $O_2(A_2B_2C_2\ldots)$ are projective.

(3) If two plane figures, S_1 and S_2, are such that to every range or pencil in the one corresponds a range or pencil of the same cross ratio in the other, then S_1 and S_2 are projective.

Let $(A_1B_1C_1 \ldots)$, $(A_2B_2C_2 \ldots)$ be a pair of corresponding ranges in S_1 and S_2, and O_1 and O_2 a pair of corresponding points.

Project $O_2(A_2B_2C_2 \ldots)$ into $O_1(A_1B_1C_1 \ldots)$.

If l_2, the line of S_2 corresponding to any line l_1 in S_1, projects into l_3, l_1 and l_3 will intersect on A_1B_1.

For if l_1, l_3 cut A_1B_1 in D_1, D_3 respectively, and if l_2 cuts A_2B_2 in D_2,

$$\{A_1B_1C_1D_1\} = \{A_2B_2C_2D_2\} \text{ (by hypothesis)}$$
$$= \{A_1B_1C_1D_3\}.$$

Hence D_1, D_3 coincide.

Since corresponding lines of S_1 and S_3 intersect on A_1B_1,

$$S_1 \text{ is projective with } S_3, \quad \text{and therefore with } S_2.$$

The theorem in (3) combined with the harmonic property of the complete quadrangle proves that any two quadrangles are projective.

II. By means of (3) we can prove that any non-singular transformation

$$\left.\begin{aligned} x &= p_1\xi + q_1\eta + r_1\zeta, \\ y &= p_2\xi + q_2\eta + r_2\zeta, \\ z &= p_3\xi + q_3\eta + r_3\zeta, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

is a projective transformation, and, conversely, that any projective transformation can be expressed analytically by the equations (A).

An analytical proof that any quadrangle $A_1B_1C_1D_1$ is projective with any other quadrangle $A_2B_2C_2D_2$ is obtained as follows:

Take the diagonal triangle of $A_1B_1C_1D_1$ as reference triangle for the first figure and the triangle $B_2C_2D_2$ as the reference triangle for the second. If (f, g, h) are the actual coordinates of A_1, the coordinates of B_1, C_1, D_1 are proportional to $(-f, g, h)$, $(f, -g, h)$, $(f, g, -h)$.

The coordinates of A_2, B_2, C_2, D_2 are (ξ_1, η_1, ζ_1), $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

Since the point $(-f, g, h)$ becomes $(1, 0, 0)$,

$$-\lambda f = p_1, \quad \lambda g = p_2, \quad \lambda h = p_3.$$

Similarly

$$\mu f = q_1, \quad -\mu g = q_2, \quad \mu h = q_3;$$
$$\nu f = r_1, \quad \nu g = r_2, \quad -\nu h = r_3;$$

where λ, μ, ν are parameters arising from the fact that the ratios of coordinates determine the points.

To complete the determination of the transformation (A) it is necessary to find λ, μ, ν.

Since (f, g, h) transforms into (ξ_1, η_1, ζ_1),

$$-\lambda\xi_1 + \mu\eta_1 + \nu\zeta_1 = 1,$$
$$\lambda\xi_1 - \mu\eta_1 + \nu\zeta_1 = 1,$$
$$\lambda\xi_1 + \mu\eta_1 - \nu\zeta_1 = 1.$$

The discriminant of this set of equations for λ, μ, ν is $2\xi_1\eta_1\zeta_1$.

Since A_1 is not on the sides of the reference triangle $B_1C_1D_1$ the discriminant does not vanish.

Hence λ, μ, ν are uniquely determined.

That is:—

There is one and only one projective transformation that will transform the quadrangle $A_1B_1C_1D_1$ into $A_2B_2C_2D_2$.

The University of Queensland,
Brisbane, Oct. 12, 1923. H. J. Priestley.

725. [V. 1. a.] Mnemonics are commonly an appeal against logical principles. The following are intended as an appeal against arbitrary rules *to* logical principles. Perhaps they are not too flippant for publication.

"Please, Sir; do you add the Brokerage or subtract it?"

The Broker must have eggs and ham,
And mend the little brokers' pram.
How can he pay his fare to town
By *giving* you the odd half-crown?
So pay some more without a shock
When Mr. Broker buys you stock.
And when the Broker helps you sell,
And takes his nibble off, say, "Well,
You've earned it": don't say anything that
might hurt his feelings.

"Please, Sir; is his income 5 per cent. of the Stock he bought or of the price he paid for it?"

The Railway neither knows nor cares
How you obtained your blessed shares,
Bought in a pawn-shop second-hand
Or found in Joe's umbrella-stand.
Why should it pay you 5 per cent
On what *you say* that you have spent?
And if you stole them, I'm afraid
Your fortune won't be quickly made
By 5 per cent. of *what you paid.*

ELPIS.

726. [V. 1. a. μ.]. *To find the position of P within a given triangle ABC, so that* $l.PA+m.PB+n.PC$ *shall be a minimum* (l, m, n *being known ratios*).

Take three strings each of length d inches; and tie together in a knot their three ends, one of each string. Pass the strings respectively through three eyelet holes (A, B, C) in a thin horizontal lamina; and to their free ends attach weights of l, m, n pounds.

Then the vertical distances of the three weights below the lamina are (in inches) $d-PA$, $d-PB$, $d-PC$; and that of their centre of gravity

$$\{l(d-PA)+m(d-PB)+n(d-PC)\}/(l+m+n).$$

When there is equilibrium, this latter distance is a maximum, and consequently $l.PA+m.PB+n.PC$ is a minimum.

The knot is then acted upon by three tensions of l, m, n pounds in directions PA, PB, PC; and for equilibrium $\sin BPC : \sin CPA : \sin APB = l:m:n$ (Lami).

When $l=m=n$, then angles $BPC=CPA=APB=120°$ (Fermat's point), provided, of course, the greatest angle A is not more than 120°.

R. F. DAVIS.

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254. Monge, who was very fond of the Marseillaise, used to sing it every day at the top of his voice before sitting down to a meal.—*Arago.* (Per Prof. Genese.)

255. To show their gratitude to a colleague, whose scientific fame had already spread over the whole of Europe, the Cambridge professors nominated Newton, it must be admitted by only a small majority, to represent them in the parliament of 1688.

This parliamentary career was without distinction. It is said that during its whole course, he opened his lips but once—to ask the door-keeper to shut a window, the draught from which might be giving a cold to a speaker addressing the House.—*Arago.* (Per Prof. Genese.)

REVIEWS.

Lectures on Cauchy's Problem in Linear Partial Differential Equations. By J. HADAMARD. Pp. viii + 316. 15s. net. 1923. (Per Oxford University Press.)

There is, fortunately, no longer a thick hedge between the domains of Pure and Applied Mathematics. It is recognised that any branch of mathematics, however pure, may become polluted by being applied to physics or chemistry; and that a pure mathematician may even prefer to find his problems hidden within the hideous portals of such a subject as the conduction of heat or the motion of ants. It is not a distinction in the subject but only in the direction in which the interest lies—whether it is in the facts or in the logic uncovered in investigating the facts. Even the theory of numbers is applied mathematics if the real interest is to discover the properties of actual numbers rather than their logical relations as determined by their mode of construction.

The book under review supplies a good illustration; for although the Cauchy problem arose out of physics it is here considered evidently because of the delightful and delicate points in logic which arise out of it. That is to say, Professor Hadamard is a Pure Mathematician, who, for all that, is not disdainful of physical concepts—and these, it may be added, as pure concepts, are a part of Pure Mathematics.

The problem handled by Professor Hadamard has been "solved" many times; but the solution has usually had little to do with the problem as it actually ought to be set. The real interest arises in discussing what should and what should not be given.

Given, for example, the differential equation

$$Ar+2Hs+Bt+2Gp+2Fq+Cz=0,$$

where A, H, B, G, F, C, are known functions, z the unknown function of x and y, and r, s, t, p, q have the usual meanings, the real problem is *not* to find the general solution of this equation by itself, but to find the solutions (if any) of this equation together with given "boundary conditions" such as the given values of z and q on a given line parallel to the y-axis.

The interest is also in finding what boundary conditions are mutually consistent and consistent with the equation. The fact, shall we say the remarkable fact, is that this fascinating study of logical hypotheses is at the same time exactly the study demanded by the practical physicist. It is this study to which the lectures are devoted, and it is hoped that all mathematicians will give themselves the pleasure of reading the book and appreciate the manner in which even practical demands are used as guides to logical truth.

Only one criticism may be breathed, that the book is somewhat discursive. It is not a treatise which progresses steadily. It is rather an attack—always with one objective—but with many flanking movements. This, however, adds to the freshness and value of the book. The reviewer would, personally, not have wished it put into more strait a jacket.

The general problem is as follows:

Given a surface S, $G_n(x_1, x_2, \dots x_n)=0$, and a differential equation for u,

$$\Sigma A_{ik}s_{ik}+\Sigma B_i p_i+Cu=0,$$

$$p_i=\partial u/\partial x_i, \quad s_{ik}=\partial p_i/\partial x_k;$$

Cauchy's problem for that equation with respect to the surface S will consist in finding a solution satisfying at every point of this surface two conditions such as

$$u=u_0, \quad \partial u/\partial N=u_1,$$

N being a direction given arbitrarily at each point of S, but not tangent to it, u_0, u_1 being given numerical values at each point of S, the latter being called Cauchy's data for the present case.

In the case of waves of sound, $n=4$, $x_4=t$, S is the 3-spread, $t=0$, and Cauchy's conditions would be that

$$u(x, y, z, 0), \quad [\partial u/\partial t]_{t=0}$$

be given functions of x, y, z.

Cauchy's fundamental theorem is developed by means of "dominant" functions, and the "characteristics" appear as leading to the exceptional case. There then arises the "characteristic conoid," $\Gamma=0$, which has a given point $a(a_1, a_2, \ldots a_n)$ as a conic point, whose surface is singular for the given differential equation and which satisfies the first order equation

$$A(\partial G/\partial x_i, x_i)=0,$$

where A is the quadratic form composed from the coefficients of s_{ik} in the differential equation.

This conoid determines an "elementary solution" which takes the place of $1/r$ in ordinary potential theory,

$$r^2=\Gamma=(x-a_1)^2+(x-a_2)^2+(x-a_3)^2=0$$

being the equation of a "conoid," in fact of a cone.

Professor Hadamard now diverges to define a type of improper integral which is of special value in handling inverse powers of G which arise in the solution in terms of the given boundary conditions.

The rest of the book deals with proofs of the solutions obtained in terms of the conoid and with boundaries which coincide partially with characteristics. There are also numerous illustrations from the classic problems. The case of an odd number of variables (*e.g.* x, y, t in cylindrical waves) is treated first, because the index of the inverse power of Γ which appears in the integral is then not an integer, and the improper integrals can be used. The case of an even number is then considered by what is called by the writer, rather unfortunately perhaps, the "method of descent." This is nothing else than a solution of the corresponding equation with an additional auxiliary variable. There is another "method of steepest descent," used in obtaining asymptotic expansions for contour integrals containing terms of type $\exp(inz)$, where n is large. But there will probably arise little confusion between the two phrases.

In conclusion, the author is to be congratulated on his nice use of English, and readers will second the author in his thanks to Mr. Walsh and Mr. Murray for their help in this respect.

We heartily recommend this book to those who like to *enjoy* their Mathematics.

The University, Sheffield. P. J. DANIELL.

The Psychology of Arithmetic. By E. L. THORNDIKE. Pp. xvi+314. 9s. net. 1922. (The Macmillan Co.)

The Psychology of Algebra. By E. L. THORNDIKE and Others. Pp. xi+483. 10s. 6d. net. 1923. (The Macmillan Co.)

A careful reading of these two books leaves the reviewer in some doubt, not as to his personal attitude, but as to what he had better say about them. A cursory glance would have left a very unfavourable impression, but a closer perusal shows that there is much in the books that should be welcome to British readers. The psychology by which British teachers have effected reforms in mathematical teaching has consisted largely of commonsense observation of pupils, and a general feeling, no less positive because not always precisely defined, for the aims of school mathematics. But we have been unwise in attaching so little importance to positive psychological experiment. A glance at the bibliographies of these books reveals a striking contrast between our relative indifference and the enormous interest displayed in the psychology of school subjects in America. It is an interesting question, why, with all this research, there is so much in American educational writings (in these two books for example) that appears retrograde to British readers. Why is so large a proportion of these two books devoted to the relatively mechanical aspects of learning? The answer is that American writers make a determined attempt to base their principles of teaching upon the results of psychological experiment, and such experiment has unhappily led many of them to a very mechanical kind of psychology. British teachers are, however, too lukewarm in their recognition of the experimental point of view in educational psychology, and we unhesitatingly invite them to read these two books, partly because there is much good psychology in them mixed up with

the bad, and partly because the definite psychological standpoint they represent, if definitely wrong, is yet instructive because it *is* definite.

Prof. Thorndike is the leading exponent of a school of educational psychology which is fairly described in the Preface to the *Arithmetic* thus: "We now understand that learning is essentially the formation of connections or bonds between situations and responses, that the satisfyingness of the results is the chief force that forms them, and that habit rules in the realm of thought as truly and as fully as in the realm of action." Or again, in the *Algebra*: "... we have resolutely applied to the pedagogy of algebra the facts and principles which recent work in the psychology of learning has established. ... it emphasises the dynamic aspect of the mind as a system of connections between situations and responses; treats learning as the formation of such connections or bonds or elementary habits; and finds that thought and reasoning —the so-called higher powers—are not forces opposing these habits but are these habits organised to work together and selectively." Here, and in a score of other quotations that might have been given, two noteworthy and allied factors are either ignored or rejected. There is no reference to "Purpose" as a fundamental psychological category; and the insistence upon the notion that reasoning is, not "a somewhat magical power or essence," but "an organisation of habits," something which can be described in terms of habit only, suggests that "Intelligent" (dare we use the term), as an adjective qualifying behaviour that is essentially not capable of mechanical interpretation, is regarded as an unnecessary term. We should not mind the Americans (of the Thorndike school) believing this psychology if only they would keep it for armchair use and not try to put it into practice in their schools! It is a psychology that cuts very deep into educational practice. Only two objections can be here considered. We suggest that the formula of response to stimulus, the response being made because of its "satisfyingness," is totally inadequate, and to a great extent meaningless, as a description of the process of learning. And we submit that it is not frivolous to ask, if reasoning is "an organisation of habits," *what* organises them and *why* does it organise them? The answer, put as briefly as possible, is that it is "we" who organise our habits, and that the process by which our knowledge grows into an orderly and meaningful structure is marked by a degree of conative unity throughout. The alternative to this description seems to lead to an endless regression. Our habits are organised by "higher" habits, those higher habits by higher habits still, and so on. Criticism of the Stimulus-Response formula can be very inadequately given here. If the "satisfyingness" of a response is to be the cause of the response it must precede the response and somehow qualify the stimulus. There is no means in Thorndike's psychology by which this, so to speak, anticipatory satisfyingness can be achieved. But if, on the other hand, the stimulus and response are bound together by conation there does not seem to be quite so much mystery about the fact that the response (" selected " from a number of possible alternative responses) is "relevant" to the stimulus; that is, the concept of conation enables us to give an intelligible description of the growth of organised knowledge, by relevant responses, which the mechanistic psychology of Thorndike is unable to give. It may also be pointed out that Thorndike, in the matter of Stimulus and Response, appears to reason in a circle. For he defines a "satisfying state of affairs" as one which we do nothing to avoid. The cause of a response (*i.e.* of the selection of one rather than another response) is its satisfyingness. Thus all that we know is that we make a response because we do nothing to avoid it!

Chapter I. of the *Arithmetic* gives an admirable analysis of arithmetical "abilities," which is continued, with emphasis on computation rather than the solution of problems, in Chaps. III. and IV. In Chap. II. are described some of the Standard Tests in arithmetic (Courtis, Woody, Thorndike) which have been evolved with great thoroughness in America. Though these tests leave much to be desired, and are in places suggestive of a kind of teaching of which British teachers increasingly disapprove, they are excellent in their way, and teachers have here a compact account of them which they will not find in any other book. (A fuller account of Standard Mathematical Tests

will be found in the *Report on the Reorganisation of Mathematics in Secondary Education*, just issued by the Mathematical Association of America.) The emphasis on Habit leads naturally to an exaggerated opinion of the importance of "drill," and 40 pages of the book (Chaps. V. and VI.) are devoted to this topic. In calculation, the book seems to suggest, pupils should learn the arrangement of their (written) work by imitation, without any explanation of why and wherefore, the rule being verified empirically—the pupils see that it gives the right answer in all cases. Chapters IX. and X. contain an exposition of Thorndike's general educational psychology, and should perhaps be read before the earlier chapters by those to whom the American Stimulus-Response psychology is unfamiliar.

With the exception of Chapter II., on the Uses of Algebra, which contains a good deal of sheer nonsense, there are few chapters in the *Algebra* that do not contain valuable information and useful suggestions. As especially valuable should be mentioned Chapters VI.-IX., on "Algebraic Abilities." These chapters are not unexceptionable, but they contain much that is not to be found in any British books. The chapters bearing more directly on practical teaching problems show that Prof. Nunn's books have had considerable influence upon Thorndike, and that is perhaps the best feature of the book! What strikes one, however, is that much of the psychology of the book really does nothing to strengthen convictions—such as that the formula and graph should be the basis of early work in algebra—which we had before reading the book. On the other hand, psychology leads the author to some conclusions with which we cannot agree. On the principle "Do not form two or more habits where one will do as well," we are advised to omit all methods of solving quadratic equations except by means of the formula. This surely makes it hardly worth while teaching quadratic equations, which have few applications at all in school. The value of such equations, we suggest, lies in the illustration they supply of equations with more than one root (which only the graphical approach makes clear to many pupils), and in the intrinsic interest of such methods as completing the square (illustrated by a geometrical construction in which an *actual square* is "completed"); we do *not* teach such equations just in order to develop the comparatively unimportant habit of solving them. Chapter VI. presents a useful summary of Standard Tests in algebra, and the chapter on "The Equation" has some useful suggestions. Drill in algebra receives the extensive treatment we expect, in Chapter XIII.; a great part of this chapter is not likely, we hope, to appeal to readers over here. The chapter on "Interest in Algebra," like the corresponding chapter in the *Arithmetic*, is based largely on censuses of pupils' preferences for one kind of topic rather than another. The Stimulus-Response psychologists are naturally not happy in their treatment of "Interest," which really has no meaning apart from conation, and the latter does not enter in a recognisable form into their psychology. Thus the chapter on "Interest" in the *Arithmetic* is padded with irrelevant discussion of how to relieve eyestrain in written work. And too much is made of the interest the pupil has in doing things just because he *can* do them: which is natural, for that is the only kind of interest an intelligent pupil *could* take in much that Prof. Thorndike would have him do. In my own schooldays, I fear I must confess, I took comparatively small interest in getting my answers right; as soon as I had seen how the problem was to be solved the greater part of the interest vanished. That was very wrong of me; perhaps it was because I set a greater value upon intelligence than upon habit!

In spite of the defects, of psychology and methodology, with which these two books teem, we strongly recommend them. They present an experimental method of approaching school problems which is none the less valuable because in the hands of some inquirers it has led to false conclusions; and it is an excellent exercise to read the books, if only in order to disagree with them.

E. R. Hamilton.

Statique et Résistance des Matériaux. By Paul Montel. Pp. 273. n.p. 1924. (Gauthier-Villars.)

The methods employed in this book are almost entirely graphical. It is based on lectures given at the École des Beaux-Arts, where the pupils, as

the author remarks in his preface, are accustomed to use their eyes as well as their pencils and rulers so that graphical methods should specially appeal to them.

Force diagrams and funicular polygons appear early, and much ingenious work is done with their help throughout the book. The most important propositions in this part of the subject are a method for drawing funiculars through two or three given points, and the proof that a curve whose second derivative is a given curve may be obtained by taking the given curve as a curve of loads and finding a corresponding funicular. When Bending Moments and Deflections are reached, the ordinary results in the theory of the Strength of Materials are very neatly deduced, since the second derivative of the Bending Moment gives the density of loading at any point of a beam, while the second derivative of the deflection is proportional to the Bending Moment.

The deflections in the ordinary standard cases of loaded beams are deduced from the funiculars with the help of nothing more elaborate than a knowledge of the positions of the centroids of rectangles, triangles and parabolas. But the work requires some very clear thinking.

The ordinary analytical way of dealing with deflection, by solving the equation $\frac{d^2y}{dx^2}=f(x)$, has the great advantage that all the problems are solved in exactly the same manner, and depend on no harder algebraical operation than the integration of $(x-a)^n$ with respect to x; while the manipulation of the special conditions obtaining at the points where loads occur inculcates many useful lessons in the calculus, though perhaps this method is rather bludgeon-like when compared with the rapier play of graphical work.

The work is an admirable compendium of graphical statics; it is well written, and on the whole easy to read. There are several places where reference to previous propositions by quoting the number of a paragraph would be of great help to the reader. A few of the figures in which there are many lines would be clearer if the lines representing the original forces were thickened a little so as to stand out more clearly from the lines representing the funicular polygon and its construction lines. In the work on the deflection of beams, there is also a sameness about the diagrams which is not usual in similar books published in this country. W. M. R.

Synopsis of Applicable Mathematics, with Tables. By L. SILBERSTEIN. Pp. xii + 250. 1923. (Bell & Sons.)

This is a reprint of the useful book published in 1922 under the title "Bell's Mathematical Tables," and an account of the contents has already appeared in the *Gazette* (vol. xi. p. 238). We have not changed our opinion of the value of Dr. Silberstein's work, we welcome the change of title, and now we need only record regret that correction of the harmless misprints has not yet been possible.

A Treatise on the Integral Calculus, with Applications, Examples, and Problems. By J. EDWARDS. Vol. II. Pp. xvi, 980. 50s. 1922. (Macmillan.)

One to whom twenty-odd years ago Mr. Edwards' books for beginners were magic casements, cannot without regret see their author beyond hope of rescue in the perilous seas. Yet it is indisputable that in this volume Mr. Edwards is again and again out of his depth, though he seldom even warns the reader that the waters are deep.

Since the first volume of the treatise was published, the claim that principles are expounded has been abandoned. We are fully in sympathy with the author's aim as now expressed. Dexterity in the processes of integration is invaluable to the mathematician, be he physicist or analyst, and is not acquired subconsciously during a study of fine points in the theory of convergence or in the theory of aggregates. Not only is practice essential, but the student is more likely to be interested in examining conditions for the inverting of a double-limit operation if he has a wide experience of the powers that inverting confers. Unfortunately, it is impossible to regard this book only from the standpoint of formal manipulation; formal transformations must have a

theoretical basis, and instead of giving references, the author either ignores the possibility of objection, or relies on general assertions which in many cases are careless, misleading, or false.

One third of the book is concerned with real definite integrals; there is a multitude of results, which, of course, are true results, and generous practice is afforded in rough-and-ready methods of evaluation. Our complaint is not that tentative methods are used, but that the reader is not taught that some steps are safe and some dangerous. For example, six investigations are given of the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx;$$

each involves some step which is in need of justification, but no attention is drawn to the crucial steps; indeed, one proof sets out as if

$$\lim_{h\to 0,}\, h\sum_{n=1}^{\infty} f(nh)$$

was a *definition* of

$$\int_0^\infty f(x)\,dx.$$

Now the practical man is right in bringing to bear on an integral which presents itself any device which he knows to be sometimes successful; no one wants him to waste time, his own or anybody else's, in justifying a process before finding out whether the process will be effective, and for this reason Mr. Edwards' chapters will be of far greater service to him than any mere table of results. But it is of the utmost importance that the practical man should *recognise* the steps at which fallacies can creep in; having reached a conclusion which would be satisfactory, he must be able to submit crucial steps to expert examination, before reputation or life is staked on his results. It is here that Mr. Edwards fails entirely to help the very man who is most dependent on him.

The complex variable must enter sooner or later, even if real integration is the object of study. It is surprising that Mr. Edwards should expect a student who has reached the middle of this second volume to need a description of the Argand diagram and a proof of De Moivre's theorem; in fact, acquaintance with the exponential function of the complex variable has been assumed almost from the beginning of the treatise, and is taken for granted without further explanation. With mention of functionality and continuity the author's limitations become conspicuous. The fantastic suggestion (p. 379) that ϵ is to be "arbitrarily chosen smaller than anything that can be conceived beforehand, however small," is not likely to do harm, but in § 1226 we read, correcting an obvious misprint: "We may put this condition (for continuity) into . . . another form, . . . viz. that for any assignable positive infinitesimal ϵ, however small, which may be chosen beforehand, it may be possible to choose another infinitesimal δ of no higher order of smallness than ϵ, so that if $x \sim x_0 < \delta$, then will $f(x) \sim f(x_0) < \epsilon$." The reference to order of smallness is not accidental, for similar phrases occur in §§ 1225, 1227, 1230. It is safe to assert that no student can attain to a real understanding of continuity as long as he is under the impression that orders of infinitesimals have anything whatever to do with the matter. If $f(x)$ is the real cube root of the real variable x, we must take $x \sim 0 < \epsilon^3$ to secure $f(x) \sim f(0) < \epsilon$. Would Mr. Edwards teach that the function is discontinuous at 0, or that ϵ and ϵ^3 are infinitesimals of the same order? Of course, this question does not belong to the integral calculus. That would have been an excellent reason for saying nothing about it; it is no excuse for a treatment that is intrinsically indefensible.

There are no subtleties in the account of Cauchy's theorem. The analysis of loops and the account of the relation of periods to branch points are good, at the level indicated by diagrams in which pins are pictured as sticking up from all the branch points to prevent threads from passing across. The examples of the calculus of residues cover the usual ground. It is the formal theory of the standard elliptic functions, not the functional theory of doubly-periodic functions, which interests Mr. Edwards. He deduces the double periodicity of $\operatorname{sn} u$ and $\wp u$ from the branch points of the corresponding integrands, but the greater part of the hundred pages on elliptic functions consists

of straightforward deductions from addition theorems, of reductions to standard forms, and of similar work, in all of which the author is a safe guide.

A section on the calculus of variations is traditional in English treatises on the integral calculus, but there seems little reason for Mr. Edwards to have followed tradition. The problems lead to differential equations, not to quadratures, and the bookwork might have been written thirty years ago by a writer out of touch with continental mathematics; the "bibliography" on p. 691 is really pathetic. The chapter on Fourier's theorem is not out of place, since it is to quadratures that the practical applications reduce; the chapter is redeemed slightly by careful diagrams to illustrate for Poisson's and Dirichlet's integrals the concentration of effect into a fraction of the range of integration, but if we say that there is no reference to any mean value theorem, the difficulty of taking the analysis seriously will be realised.

The sections on geometrical mean values and probabilities are comprehensive and, of course, uncritical. The law of inverse probability is, quite properly, merely accepted as a basis for calculation, but a reference to Keynes' logical discussion might have been inserted. One method of approaching the problems connected with the random placing of a given rigid contour on a plane with parallel rulings presents the curious feature that the initial hypothesis, not intrinsically plausible, is definitely contradictory to the conclusions drawn from it, and is therefore certainly false! A satisfactory alternative is given also, but we cannot help wondering at the temerity of an author who writes on this subject without, as it seems, knowing more of Czuber's treatise than that Sylvester refers to it.

A chapter is given to the theory of errors, and one to the theorems of Stokes and Green and to harmonic analysis. The work on this last topic is most disappointing; since it is formal, there is no reason why it should not have been well done, but if thirty-three pages out of forty-five are on Legendre's coefficients, and no other special kind of harmonic receives any attention, the section can fairly be described as ill-balanced. The book ends with a chapter of notes.

Throughout the work the printing is admirable, and a word of thanks is due for the many excellent diagrams; those illustrating the change of variables in a multiple integral are beyond praise. For the rest, we have a valuable collection of formulae, which would be still more valuable if the table of contents was more elaborate. If the theoretical fabric is unsound, students may at least be exercised in pointing out and repairing the cracks. After all, it is idle and unfair to complain because a book displays precisely those qualities which are associated by common consent with the name of its author, and perhaps all that is reasonable to say is that if Mr. Edwards had secured the collaboration of a mathematician of complementary tastes to replace inaccurate groundwork by references to books of the standard of *Modern Analysis*, and to indicate the unexamined assumptions that are everywhere made, a notable treatise might have been produced. E. H. N.

The Properties of Matter. By B. C. M'Ewen. Pp. vii + 316. 10s. 6d. net. 1923. (Longmans.)

There is no real reason why a book with the title *The Properties of Matter* should not include the whole of physics and chemistry, but as a matter of fact English writers generally use this heading for such branches of physics as gravitation, elasticity, capillarity, diffusion and viscosity. However, Prof. M'Ewen's scope is rather wider, and he devotes more than half his space to thermodynamics. His intention is to meet the requirements of Indian students, particularly those studying for the B.A. of the University of Madras. The work appears to be well done, and the numerous diagrams (there are 137) make the arguments very clear. H. T. H. Piaggio.

Binomial Factorisations. By Lt.-Col. Allan J. C. Cunningham, R.E. Vol. I. Pp. xcvi + 288. 15s. Vol. IV. (Supplement to Vol. I.). Pp. vi + 160. 5s. 1923. (Francis Hodgson.)

For many years Lt.-Col. Cunningham has been engaged in searching for the prime factors of numbers of the type $a^n \pm 1$. The results of his investigations

are announced to appear in seven volumes, of which the first two are before us. Vol. I. contains a variety of tables: to indicate their general content it will be sufficient to fix attention on numbers of the type $M=x^4+y^4$.

The congruence $y^4+1\equiv 0 \pmod p$ has four integral roots when p is a prime of the type $8n+1$, or a power of such a prime. Col. Cunningham first tabulates the four y's for each p less than 10^5. A reference to the table shows what primes p divide y^4+1 when y is given. Making use of this table the author attempts to factorise $N=y^4+1$ for all values of y up to 1001, and actually expresses each N as a product of prime factors as far as $y=319$. For values of y from 320 to 1001 all the factors less than 10^5 are cast out and many N's are factorised completely. In other cases a partial decomposition is effected, the remaining factor ($>10^{10}$) being either prime or a product of primes exceeding 10^5,

e.g. $$907^4+1=2\,.\,240797\,.\,14042233,\text{ all primes,}$$
$$978^4+1=17\,.\,1217\,.\,2473\,.\,17881,$$

whereas $1001^4+1\equiv 0 \pmod{2\,.\,17\,.\,17}$, but contains no other factor less than 10^5.

Next follows a table giving the complete factorisation of M (or $2M$) $=x^4+y^4$ whenever $M<9,10^6$, results which could also be obtained from the published factor tables. A few integral solutions of

$$x^4+y^4=x'^4+y'^4$$

are tabulated too, *e.g.* $x=1623$, $y=3494$, $x'=2338$, $y'=3351$. Finally are given complete lists of the prime values of x^4+y^4 and $\frac{1}{2}(x^4+y^4)$ less than 10^7, also of the prime values of y^4+1 up to $y=312$ and of $\frac{1}{2}(y^4+1)$ up to $y=399$.

Theoretically the table is unsatisfying in one particular. If y_1 is any root of the congruence

$$y^4+1\equiv 0 \pmod p,\text{ with } p\equiv 1 \pmod 8,$$

the four roots are

$$y_1,\quad y_2\equiv y_1^3,\quad y_3\equiv y_1^5,\quad y_4\equiv y_1^7 \pmod p.$$

The table presents y_1, y_2, y_3, y_4 in order of numerical magnitude, and a calculation is necessary to arrange them in the theoretically correct order y_1, y_1^3, y_1^5, y_1^7.

Col. Cunningham has spared no pains to ensure accuracy in the results he gives. Results of the type contained in *Binomial Factorisations*, when recorded once, are made available for all time, and the scientific world is deeply indebted to the author for the labour he has incurred in calculating the tables which he now presents to it. W. E. H. B.

A General Text-Book of Elementary Algebra. By E. H. Chapman. Pp. vii+512. 7s. 6d. net. (Blackie.)

This volume covers the syllabus for the London B.Sc. Part I. consists of elementary algebra up to Quadratic Equations. Part II., from which the teacher is intended to make a selection, is also quite elementary. The treatment is on modern lines, but is not so different from that in other text-books as to make it clear why it was necessary to publish another. Part III. ranges over the usual subjects from Variation to the (elementary) Binomial Theorem, and then passes on to Limits, Convergence, the Binomial, Exponential, and Logarithmic Theorems, and the theory of Complex Numbers. It is to be regretted that Dr. Chapman should have seen fit to write on these subjects at all. The essential defect in his treatment of Limits is its vagueness, but it is not so vague that one cannot pick out one or two of the mistakes. For example, it is wrong to deduce $f_1-f_2<b_1-b_2$ from $f_1<b_1$ and $f_2<b_2$. Also, in enunciating $\lim(f/\phi)=(\lim f)/(\lim \phi)$, it *should* be stated that $(\lim \phi)$ is not zero; on the other hand, it is *not* necessary to assume that a is not zero in the theorem $\lim\limits_{h\to\infty}(a/h)=0$. The chapter on Convergence is perhaps worse than that on Limits. In the "proof" of the convergence of $1-\frac{1}{2}+\frac{1}{3}-\ldots$, the existence of $\lim s_n$ is said to follow from $0<s_n<1$, or possibly it is meant to be derived from $0<s_{2n}$ and $s_{2n+1}<1$; it is also assumed, near the beginning of the proof, that $1-\frac{1}{2}+\frac{1}{3}-\ldots$ is a convergent series. An oscillating series is defined as a series such as $1-1+1-\ldots$, which may have a sum 0 or 1 *according as infinity is even or odd*. As might be expected, the applications of the theory

of Convergence to the Binomial and Exponential Theorems are entirely misleading.

In the short chapter on complex "quantities," the geometrical representation is given. The examples set at the end of the chapter show a complete misunderstanding of the logic of the subject. For one does not *prove* that $a=c$ and $b=d$ follow from $a+bi=c+di$; nor is it possible for the sum or product of two complex numbers to be a real number.

At an earlier stage of the book the author makes the unfortunate statements that $\sqrt{-5}$ has no *arithmetical* meaning, and that $(x-2)^2=-1$ has no *real* solutions. Just as if -5 had *some* sort of a square root and the equation *some* sort of a solution! At that stage, the student can only regard -5 and -1 as negative numbers, and he believes that a negative number has *no square root at all*. He is right. Why deceive him? A. Robson.

MANCHESTER AND YORKSHIRE BRANCHES.

A Joint Meeting of the Manchester and Yorkshire Branches of the Mathematical Association was held at Greenhead High School for Girls, Huddersfield, on Saturday, May 17. There were over 70 people present.

Mr. W. C. Fletcher, M.A., H.M.I., read a paper on the "Teaching of Geometry," which was stimulating and suggestive, and gave rise to a short helpful discussion.

Prof. P. J. Daniell, D.Sc., of Sheffield, then gave an account of "American Reforms in the Teaching of Algebra." A hearty vote of thanks was accorded to both speakers, and the meeting adjourned for tea. The thanks of the Association were conveyed to Miss Hill, Headmistress of the Greenhead High School, and to Mr. Thornber, Secretary of the Huddersfield Education Committee, for the great trouble which they had taken to make the meeting so successful.

A NEW HIGHER DEGREE COURSE IN THE HISTORY AND METHOD OF SCIENCE.

An interesting new degree course is being initiated at London University. For some time in London there have been a number of courses on the History of Science in general and the various Sciences in particular. The philosophical teachers in the University also have been taking much interest in the theoretical aspects of scientific procedure. A year ago a *Board of Studies* was established to bring the different parts of this work into correlation. A number of advanced students have been working at the subject for research degrees, and one has already taken the Ph. D.

It is now proposed to institute a degree more accessible to the ordinary post-graduate student. A scheme drawn up by this *Board of Studies* for an M.Sc. course has been accepted by the Academic Council, by which the subject of History and Method of Science is placed on the same basis as other subjects for post-graduate work. A one-year course has been arranged, after which the degree of M.Sc. will be available for candidates.

The course for the degree will be of an inter-collegiate character. Lectures are being held at University College and King's College. It is hoped that all these lectures will be given in the late afternoon or early evening, so that they will be accessible to teachers who are at work during the day. In this way those occupied in science teaching will be encouraged to pursue their studies into the more philosophical and historical departments of their subject.

The course will be divided into four parts corresponding respectively to four papers at the examination. The first part will deal with the General History of Science from the earliest times to the present day, treated along

broad and elementary lines. The second part will provide a more detailed study of the development of Science in modern times, the candidate choosing according to his previous training between the Physical and Mathematical Section on the one hand and the Biological Section on the other. The third part, entitled "Methods and Principles of Science," will deal with more purely philosophical aspects of the subject, such as the Nature of Scientific Reasoning, Conceptions of Natural Law and Scientific Ideals. In the fourth part the candidate will enter somewhat more minutely on any particular part of the syllabus as a whole which he may select as that which most appeals to him. This last section will give an opportunity to those who propose subsequently to undertake research, or who have any special interest to develop.

The course will begin next session in October, and will be open to all suitably grounded students. The degree, however, will be confined to those who have already a B.Sc. degree in the University of London or a Science degree in some other recognised University.

It is the function of the course not so much to impart a knowledge of Science in itself as to enable the candidate to grasp the general aims and nature of Science and of scientific problems, the character of scientific method, the relation of Science to other studies, and the educational value of scientific teaching. It is evident that such a course will be of peculiar value to the Science teacher.

Enquiries concerning this course should be made to the Secretaries of University College, London, or King's College, London. Dr. Charles Singer, Department of History and Method of Science, University College, London, who is Secretary of the *Board of Studies*, will also be pleased to see any prospective students by appointment, or to answer any question concerning the course. Communication should be made with him in the first instance by writing.

THE LIBRARY.

160 Castle Hill, Reading.

In memory of her husband, Mrs. Charles Godfrey is presenting practically the whole of his collection of works on mathematics and on the teaching of mathematics to this Library. The Association was a master-passion with Professor Godfrey, and in using his books we shall perpetuate as well as commemorate his enthusiastic services. A list of the volumes will be printed as soon as possible.

The Librarian reports gifts as follows:

From Mr. W. J. Greenstreet (fourth list):

N. H. Abel	Untersuchungen über die Reihe $1+\{m/1\}x+\{m(m-1)/1.2\}x^2+\ldots$ (1826; Ostwald 71)	1895
R. d'Adhémar	Equations aux Dérivées Partielles - - (Scientia 29)	1907
Aratus Solensis	Φαινόμενα καὶ Διοσημεία - - - - - - - With the Scholia ascribed to Theon. *Aratus' work is a poetical version of a work of Eudoxus; it was translated into Latin by Cicero, and is the poem from which St. Paul quoted at Athens.*	1559
C. Babbage	Passages from the Life of a Philosopher - - - -	1864
C. W. C. Barlow and G. H. Bryan	Mathematical Astronomy - - - - - -	1893
J. Bennett	Geometrical Illustrations - - - - - -	1837

JEAN BERNOULLI, JACQUES BERNOULLI, L. EULER	Abhandlungen über Variations-Rechnung (1696, 1697, 1744; Ostwald 46)	1894
B. BOLZANO	Rein analytischer Beweis eines Lehrsatzes (1817; Ostwald 153)	1905
	If $f(\alpha)$ and $f(\beta)$ have opposite signs, $f(x)$ is zero for some value between α and β.	
A. BRAVAIS	Ueber die Systeme von regelmässig vertheilten Punkten (1848; Ostwald 90)	1897
P. CARUS	The Mechanistic Principle - - - - - - -	1913
	Relativity in the light of the philosophy of science - -	1913
G. COMBEBIAC	Actions à Distance - - - - - (Scientia 30)	1910
G. H. COOPER	Octimal Arithmetic - - - - - - - - -	1902
A. COTTON	Phénomène de Zeeman - - - - - (Scientia 5)	1899
L. CRELIER	Systèmes Cinématiques - - - - - (Scientia 31)	1911
M. W. CROFTON	Notes on Mechanics - - - - - - - - -	1882
G. L. DIRICHLET	Anwendungen der Infinitesimalanalysis auf die Zahlentheorie - - - - - (1839-1840; Ostwald 91)	1897
C. F. GAUSS	Allgemeine Flächentheorie - - (1827; Ostwald 5)	1900
	Factoren Zerlegung rationaler ganzer Functionen (1799-1849; Ostwald 14)	1890
A. GIRARD	Invention Nouvelle en l'Algèbre - - - - - -	1884
	Reprint of the original edition of 1629.	
J. HADAMARD	Série de Taylor - - - - - - (Scientia 12)	1901
H. HANKEL	Oszillierende und unstetige Funktionen (1870; Ostwald 153)	1905
	This number of the Klassiker *combined papers by Bolzano and Hankel, and was edited by P. E. B. Jourdain.*	
C. G. J. JACOBI	Ueber die vierfach periodischen Functionen zweier Variabeln - - - - 1834; Ostwald 64)	1895
	Ueber die Determinanten (1841; Ostwald 77)	1896
	Ueber die Functionaldeterminanten (1841; Ostwald 78)	1896
J. L. LAGRANGE, A.M. LEGENDRE, C. G. J. JACOBI	Abhandlungen über Variations-Rechnung (1762, 1770, 1786, 1837; Ostwald 47)	1894
I. NEWTON	Tables for renewing and purchasing Leases - - -	1758
T. E. PEET	Rhind Mathematical Papyrus. Transcription, translation, and commentary - - - - - - -	1923
J. PERRY	Calculus for Engineers - - - - - - - - -	1897
M. PETROVITCH	Mécanique des Phénomènes - - - (Scientia 27)	1906
H. POINCARÉ	Théorie de Maxwell (Scientia 23. 2 editions)	[1899], 1904
G. RUTLEDGE	Topics in the Calculus - - - - - - - - -	(1923)
W. G. SPENCER	Inventional Geometry - - - - - - - - -	1892
	The second English edition, with preface by Herbert Spencer.	
R. STEELE	The Earliest Arithmetics in English - - - - -	1922
	Reprints edited with an Introduction. E.E.T.S.	
J. STEINER	Geometrischen Constructionen (1833; Ostwald 60)	1895
	Abhängigkeit Geometrischer Gestalten (2 vols.) (1832; Ostwald 82-83)	1896
J. THOMAE	Formeln der Elliptischen Funktionen - - - - -	1905
H. VUIBERT	Anaglyphes Géométriques - - - - - - -	1912
J. C. WILSON	Traversing of Geometrical Figures - - - - -	1905
H. WOODS	Æther - - - - - - - - - - - - - -	(1906)

Also offprints of papers by W. G. Brown, F. Cajori (6), G. Loria, W. H. Macaulay, G. Peano, and a number of early issues of the *Gazette.*

Other mathematical volumes of the Scientia *series and of* Ostwald's Klassiker *would be very welcome; books which are valuable in their contents, but small enough to travel by letter-post and to be read in a short time, are admirably suited to a postal lending library.*

From Rev. J. J. MILNE:

M. CHASLES	Cones and Spherical Conics. Trans. (from French into English) and annotated by C. Graves - - - - -	1841
A. H. FROST	Nasik Squares and Cubes - - - - - - -	1877
J. GRIFFITHS	Notes on Geometry of the Triangle - - - - -	1891

D. KIKUCHI	Elements of Plane and Solid Geometry (3 vols.) - - *Repairing a loss recorded in the* Gazette *for last December.*	1891
(P. MORTON)	Geometry, Plane, Solid, and Spherical (Library of Useful Knowledge)	1830
R. POTTS	Appendix to Euclid's Elements - - - - - - - *Can any member give the edition* (1845) *to which this is the sequel? The Library has the second edition* (1861), *in which most of the* Appendix *was incorporated, but a chapter 'On the theory of transversals' was not reproduced, and there was so much change between the editions that the references in this chapter are not adaptable.*	1847
S. W. WAUD	Algebraical Geometry (Library of Useful Knowledge)	1835

Also pamphlets and offprints of papers by R. J. Aley (2), R. C. Archibald (12), W. K. Clifford (9), E. Collignon, R. F. Davis (6), J. D. H. Dickson, A. H. Frost (2), R. W. Genese, G. Heppel, W. H. H. Hudson (2), M. Jenkins (2), E. Lemoine, A. Macfarlane, T. Muir, — Potomian (4), A. J. Pressland (2), A. A. Robb, T. C. Simmons (3), T. B. Sprague (2), J. A. Third, R. Tucker (5).

From Prof. E. H. NEVILLE:

I. NEWTON	Universal Arithmetick. Wilder's edition. Part 2 - - *Part 1 was already in the Library, and the two parts have been rebound uniformly.*	1769
A. A. ROBB	Absolute Relations of Time and Space - - - - -	1921

From Mr. D. K. PICKEN, copies of his books:

	Number System of Arithmetic and Algebra - - -	1923
	Number System of Arithmetic and Algebra. Part I. - *Separate issue of the first four chapters of the complete work.*	1923
	Theory of Elementary Trigonometry - - - - -	1910

and of a book written in collaboration:

W. WADDELL and D. K. PICKEN	First Trigonometry - - - - - - - - -	(1919)

The following have been bought:

J. S. BAILLY	Histoire de l'Astronomie Ancienne - - - - -	1775
	Histoire de l'Astronomie Moderne (3 vols.)	1779, 1779, 1782
	Traité de l'Astronomie Indienne et Orientale - - -	1787
M. BOCHER	Plane Analytic Geometry - - - - - - - -	1915
F. ENGEL	Grassmans Leben - - - - - - - - - -	1911
T. S. FISKE	Functions of a Complex Variable - - - - - -	1907
C. HENRY	Abrégé de la Théorie des Fonctions Elliptiques - - *Based on sections of Jordan's* Cours d'Analyse.	1895
P. KELLAND	Algebra - - - - - - - - - - - -	1876
R. POTTS	Algebra, with Historical Notes - - - - - -	1880
D. E. SMITH	Teaching of Elementary Mathematics - - - - -	1900
D. M. Y. SOMMERVILLE	Analytical Conics - - - - - - - - -	1924

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256. "Mr. Tennyson, what's the compound interest of one penny put out at the Christian era up to the present time?" was Whewell's good-natured call to attention in the lecture room while my father was reading Virgil under the desk.—*Alfred Lord Tennyson: A Memoir by his son*, 1897, i. 39.

257. . . . The expelled man ran for his life. . . . At last Whewell caught him. "Do you know who I am, sir?" said Whewell, panting. "Yes," was the answer, "Old Whistle, who made that mistake in his *Dynamics*." Thereupon Whewell . . . took him by the scruff of the neck, carried him to the great gate and shot him out like bad rubbish.—*Alfred Lord Tennyson: A Memoir by his son*, 1897, i. 39.

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(An Association of Teachers and Students of Elementary Mathematics.)

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